CONCLUSION

Port controlled Hamiltonian systems for
distributed parameter systems:

– defined on vector space of exterior differential forms
– with Dirac structure associated with exterior derivative:
Stokes Dirac structure.

Transmission line, Maxwell'equations
vibrating string, 1–D compressible fluid

Stokes Dirac structure associated with
elementary interaction between 2 physical domains
and includes power flow *through the boundary.*

Future work:

– extension to other Dirac structures: Lie–Poisson bracket
– extension to dissipative systems
– control design

1–DIMENSIONAL FLUID DYNAMICS

State–space:

mass density 1–form: $\alpha_E(t) = \varrho(z,t)\ \mathrm{dz} \in \Omega^1([a,\ b])$

velocity 1–form: $\alpha_M(t) = v(\mathrm{z},\ t)\ \mathrm{dz} \in \Omega^1([a,\ b])$

Energy functional: $\mathcal{H}[\alpha_E, \alpha_M] = \int_{[a,b]} H$ with:

$$H = \left[\varrho\ U(\varrho) + \frac{1}{2}\varrho v^2\right] \mathrm{dz} \in \Omega^1([a,\ b])$$

Gradient:

"pressure" function:

$$\delta_E H = \underbrace{\frac{1}{2}v^2}_{\text{pressure momentum}} + \underbrace{\frac{\partial}{\partial \varrho}(\varrho U(\varrho))}_{\text{enthalpy}} \in \Omega^0([a,\ b])$$

momentum function: $\delta_M H = \varrho v \in \Omega^0([a,\ b])$

Port controlled Hamiltonian formulation:

$$\begin{bmatrix} -\frac{\partial \varrho}{\partial t}(z,t) \\ -\frac{\partial v}{\partial t}(z,t) \end{bmatrix} = \begin{bmatrix} 0 & \mathrm{d} \\ \mathrm{d} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2}v^2 + \frac{\partial}{\partial \varrho}(\varrho U(\varrho)) \\ \varrho v \end{bmatrix}$$

and:

$$\begin{bmatrix} f_{a,b} \\ e_{a,b} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2}v^2 + \frac{\partial}{\partial \varrho}(\varrho U(\varrho)) \Big|_{a,b} \\ \varrho v|_{a,b} \end{bmatrix}$$

VIBRATING STRING (2)

Relation with the displacement *u*.

The strain is given by: $\varepsilon = u_z$.

Hence by :$\frac{\partial \varepsilon}{\partial t} = \frac{\partial}{\partial z} v$ and $v = \frac{p}{\mu}$ one obtains:

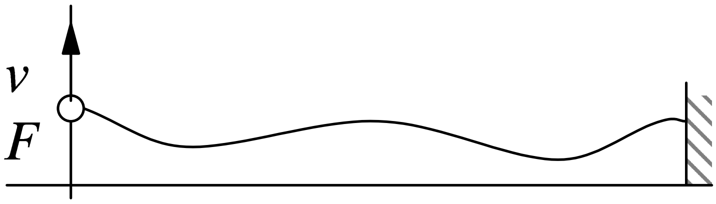
$$\frac{\partial}{\partial z}\left(\frac{p}{\mu} - \frac{\partial u}{\partial t}\right) = 0 \text{ or:}$$

$$\frac{\partial}{\partial z}\left(v - \frac{\partial u}{\partial t}\right) = 0 \; .$$

Vibrating string equation:

$$\mu \; u_{tt} - T \; u_{zz} = 0$$

VIBRATING STRING (1)



State–space of energy variables:

strain 1–form: $\alpha_E(t) = \varepsilon(z,\ t)\ \mathrm{dz} \in \Omega^1([a,\ b])$

momentum 1–form: $\alpha_M(t) = p(z,\ t)\ \mathrm{dz} \in \Omega^1([a,\ b])$

Energy functional: $\mathcal{H}[\alpha_E, \alpha_M] = \int_{[a,b]} H$ with:

$$H = \frac{1}{2}\left[T\ \varepsilon^2(z,\ t) + \frac{p^2(z,\ t)}{\mu}\right]\ \mathrm{dz} \in \Omega^1([a,\ b])$$

where: T is the elasticity modulus and is the mass density.

Coenergy variables:

stress function: $\delta_E H = \sigma = T\ \varepsilon \in \Omega^0([a,\ b])$

velocity function: $\delta_M H = v = p/\mu \in \Omega^0([a,\ b])$

Port controlled Hamiltonian formulation:

$$\begin{bmatrix} -\frac{\partial \varepsilon}{\partial t}(z,t) \\ -\frac{\partial p}{\partial t}(z,t) \end{bmatrix} = \begin{bmatrix} 0 & -\mathrm{d} \\ -\mathrm{d} & 0 \end{bmatrix} \begin{bmatrix} \sigma(z,t) \\ v(z,t) \end{bmatrix} \text{ and: } \begin{bmatrix} v_{a,b} \\ F_{a,b} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \sigma(z,t)\,|_{a,b} \\ v(z,t)|_{a,b} \end{bmatrix}$$

ELECTROMAGNETIC FIELD (2)

Boundary port controlled Hamiltonian formulation of Maxwell's equations:

$$\begin{bmatrix} -\dfrac{\partial \mathfrak{D}}{\partial t}(x,t) \\ -\dfrac{\partial \mathfrak{B}}{\partial t}(x,t) \end{bmatrix} = \begin{bmatrix} 0 & -\mathrm{d} \\ \mathrm{d} & 0 \end{bmatrix} \begin{bmatrix} \mathfrak{E}(x,t) \\ \mathfrak{H}(x,t) \end{bmatrix}$$

$$\begin{bmatrix} f_b \\ e_b \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \mathfrak{E}(x,t)\,|_{\partial N} \\ \mathfrak{H}(x,t)|_{\partial N} \end{bmatrix}$$

Power balance:

$$\frac{\mathrm{d}\mathfrak{H}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\partial N} H = -\int_{\partial N} \underbrace{\mathfrak{E} \wedge \mathfrak{H}}_{\text{Poynting vector}}$$

Assume the existence of a **current density 2–form in** N:

$$\mathfrak{J}(t) = \frac{1}{2} J_{ij}(x,\ t)\ \mathrm{d}x^i \wedge \mathrm{d}x^j \in \Omega^2(N)$$

then Maxwell's equations are augmented by **port in the domain**:

$$\begin{bmatrix} -\dfrac{\partial \mathfrak{D}}{\partial t}(x,t) \\ -\dfrac{\partial \mathfrak{B}}{\partial t}(x,t) \end{bmatrix} = \begin{bmatrix} 0 & -\mathrm{d} \\ \mathrm{d} & 0 \end{bmatrix} \begin{bmatrix} \mathfrak{E}(x,t) \\ \mathfrak{H}(x,t) \end{bmatrix} + \begin{bmatrix} -\mathfrak{J}(x,t) \\ 0 \end{bmatrix}$$

ELECTROMAGNETIC FIELD (1)

State–space of energy variables:

electric induction 2–form:

$$\mathfrak{D}(t) = \frac{1}{2} D_{ij}(x, t)\, \mathrm{d}x^i \wedge \mathrm{d}x^j \in \Omega^2(N)$$

magnetic induction 2–form:

$$\mathfrak{B}(t) = \frac{1}{2} B_{ij}(x, t)\, \mathrm{d}x^i \wedge \mathrm{d}x^j \in \Omega^2(N)$$

Coenergy variables:

electric intensity 1–form:

$$\mathfrak{E}(t) = E_i(x, t)\, \mathrm{d}x^i \in \Omega^1(N)$$

magnetic intensity 1–form:

$$\mathfrak{H}(t) = H_i(x, t)\, \mathrm{d}x^i \in \Omega^1(N)$$

Energy functional: $\mathbf{H}[\mathfrak{D}, \mathfrak{B}] = \int_N \mathbf{h}$ with:

$$\mathbf{h} = \frac{1}{2}(\mathfrak{E} \wedge \mathfrak{D} + \mathfrak{H} \wedge \mathfrak{B}) \in \Omega^3(N)$$

with constitutive relations of the medium:

$$\mathfrak{E} = \varepsilon^{-1} * \mathfrak{D}$$

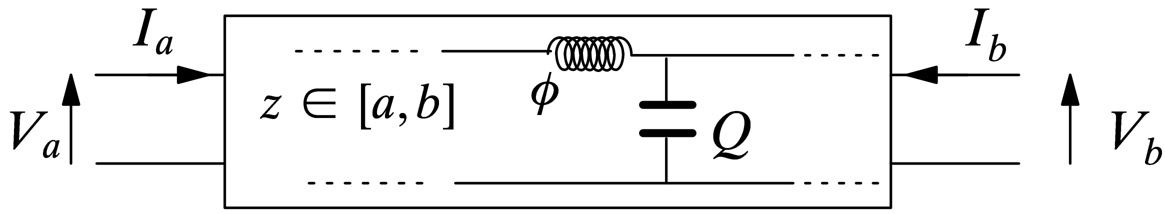
where ε is the electric permittivity tensor

$$\mathfrak{H} = \mu^{-1} * \mathfrak{B}$$

where μ is the magnetic permeability tensor

and * denotes the Hodge star product.

TRANSMISSION LINE



State–space of energy variables:

charge density: $\alpha_E(t) = Q(z, t)\ \mathrm{d}z \in \Omega^1([a, b])$

flux density: $\alpha_M(t) = \phi(z, t)\ \mathrm{d}z \in \Omega^1([a, b])$

Energy functional: $\mathcal{H}[\alpha_E, \alpha_M] = \int_{[a,b]} H$ with:

$$H = \frac{1}{2}\left[\frac{Q^2(z, t)}{C(z)} + \frac{\phi^2(z, t)}{L(z)}\right] \mathrm{d}z \in \Omega^1([a, b])$$

Coenergy variables:

voltage density: $\delta_E H = V(z, t) \in \Omega^0([a, b])$

current density: $\delta_M H = I(z, t) \in \Omega^0([a, b])$

Port controlled Hamiltonian formulation of the telegrapher's equations:

$$\begin{bmatrix} -\frac{\partial Q}{\partial t}(z,t) \\ -\frac{\partial \phi}{\partial t}(z,t) \end{bmatrix} = \begin{bmatrix} 0 & \mathrm{d} \\ \mathrm{d} & 0 \end{bmatrix} \begin{bmatrix} V(z,t) \\ I(z,t) \end{bmatrix} \text{ and: } \begin{bmatrix} V_{a,b} \\ I_{a,b} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} V(z,t)\,|_{a,b} \\ I(z,t)|_{a,b} \end{bmatrix}$$

Power balance:

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \int_{\partial[a,b]} -V\ I \quad = V(a)\ I(a) - V(b)\ I(b)$$

4. ELECTRODYNAMICS
AND THE VIBRATING STRING

The telegrapher's equations

Maxwell's equations

The vibrating string

1–dimensional compressible fluid

EXTENSION TO PORT INTERACTION IN THE DOMAIN

One may also consider port variables

distributed **in** the domain N.

Consider the extended vector space:

$$\mathcal{V}_e = \Omega^p(N) \times \Omega^q(N) \times \underbrace{\Omega^{n-q}(\partial N)}_{\text{on } \partial N} \times \underbrace{\Omega^l(N)}_{\text{on } N}$$

port variables

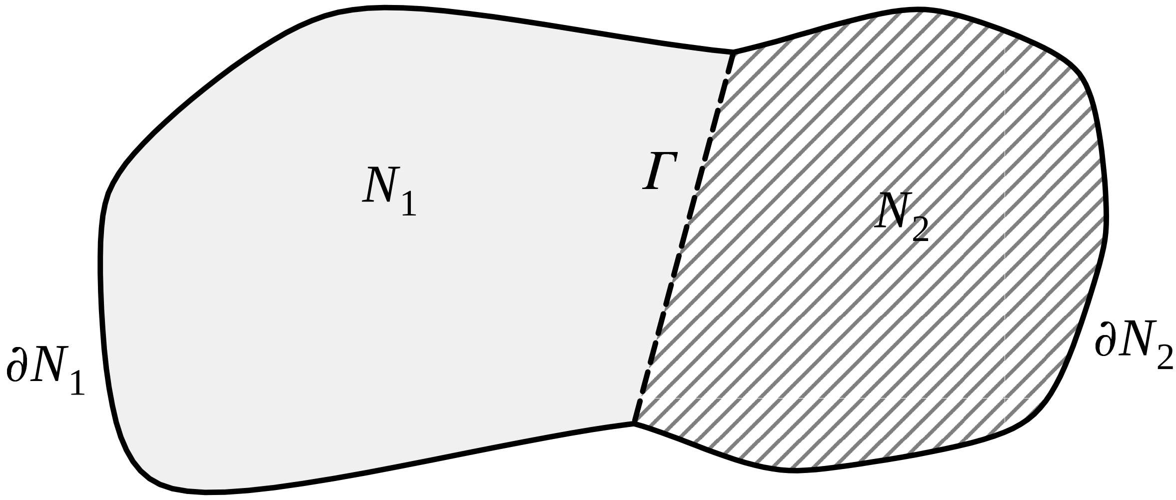
and define a Dirac structure on $\mathcal{V}_e$.

Ports on the domain N or the boundary ∂N may also be terminated by *resistive relations*, inducing dissipation.

COMPOSITION OF DIRAC STRUCTURES

Consider 2 Stokes' Dirac structures indexed by $i = 1, 2$:

$\mathfrak{D}_i$ defined on the domains N_i such that $\partial N_1 \cap \partial N_2 = \Gamma$



the composed Dirac structure $\mathfrak{D}_1 \| \mathfrak{D}_2$ is obtained on the domain $N_1 \cup N_2$ by setting:

$$f_b^1 = -f_b^2 \qquad \text{on } \Gamma$$

$$e_b^1 = e_b^2 \qquad \text{on } \Gamma$$

POWER BALANCE

The distributed parameter Hamiltonian system with power flow through the boundary satisfies the following power balance:

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \int_N < \text{grad } H \left| \left(\frac{\partial \alpha_E}{\partial t}, \ \frac{\partial \alpha_M}{\partial t} \right) \right. >$$

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \int_N \delta_E H \wedge \frac{\partial \alpha_E}{\partial t} + \delta_M H \wedge \frac{\partial \alpha_M}{\partial t}$$

$$\boxed{\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \int_{\partial N} e_b \wedge f_b}$$

Since, for $(f_E, \ f_M, \ f_b, \ e_E, \ e_M, \ e_b) \in \mathfrak{D}$:

$$\int_N e_E \wedge f_E + e_M \wedge f_M + \int_{\partial N} e_b \wedge f_b \ = \ 0$$

DISTRIBUTED PARAMETER SYSTEM DEFINED ON STOKES' DIRAC STRUCTURE

Consider the Hamiltonian density:

$$H : \Omega^k(N) \times \Omega^k(N) \times N \rightarrow \Omega^n(N)$$

and the corresponding *Hamiltonian* (total energy):

$$\mathcal{H} = \int_N H$$

Then:

$$\text{grad } H = \left(\delta_E H,\ \delta_M H\right) \in \left(\Omega^k(N) \times \Omega^k(N)\right)^*$$
$$\simeq \Omega^{n-k}(N) \times \Omega^{n-k}(N)$$

Consider state–space: $\left(\alpha_E(t),\ \alpha_M(t)\right) \in \Omega^k(N) \times \Omega^k(N)$

denoting: $f_E = -\dfrac{\partial \alpha_E}{\partial t}$ and $f_M = -\dfrac{\partial \alpha_M}{\partial t}$

and: $e_E = \delta_E H$ and $e_M = \delta_M H$

Boundary port controlled Hamiltonian system:

$$\begin{bmatrix} -\dfrac{\partial \alpha_E}{\partial t} \\ -\dfrac{\partial \alpha_M}{\partial t} \end{bmatrix} = \begin{bmatrix} 0 & (-1)^{n-k}\mathrm{d} \\ \mathrm{d} & 0 \end{bmatrix} \begin{bmatrix} \delta_E H \\ \delta_M H \end{bmatrix}$$

$$\begin{bmatrix} f_b \\ f_b \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \delta_E H\,|_{\partial N} \\ \delta_M H|_{\partial N} \end{bmatrix}$$

DIRAC STRUCTURE ASSOCIATED WITH THE EXTERIOR DERIVATIVE (2)

PROPOSITION: (Stokes Dirac structure)

$\mathfrak{D}$ is a Dirac structure

PROOF: $\mathfrak{D} \subset \mathfrak{D}^{\perp}$

Let $\left(f^{\,i},\ e^{\,i}\right) \in \mathfrak{D}$, $i = 1, 2$, then:

$$\left\langle \left(f_E^1,\ f_M^1,\ f_b^1,\ e_E^1,\ e_M^1,\ e_b^1\right), \left(f_E^2,\ f_M^2,\ f_b^2,\ e_E^2,\ e_M^2,\ e_b^2\right) \right\rangle_{\oplus}$$

$$= \int_N \boxed{e_E^1 \wedge (-1)^{n-k} \mathrm{d}e_M^2} + e_M^1 \wedge \mathrm{d}e_E^2 + e_E^2 \wedge (-1)^{n-k} \mathrm{d}e_M^1 + \boxed{e_M^2 \wedge \mathrm{d}e_E^1}$$

$$- \int_{\partial N} e_E^2 \wedge e_M^1 + e_E^1 \wedge e_M^2$$

using:

$$\mathrm{d}\left(e_E^1 \wedge e_M^2\right) = \mathrm{d}e_E^1 \wedge e_M^2 + (-1)^{n-k}\ e_E^1 \wedge \mathrm{d}e_M^2 \quad \text{etc...}$$

and Stokes' theorem:

$$\int_{\partial N} e_E^1 \wedge e_M^2 = \int_N \mathrm{d}\left(e_E^1 \wedge e_M^2\right) \qquad \text{etc ...}$$

it is shown that: $\left\langle \left(f^1,\ e^1\right),\ \left(f^2,\ e^2\right) \right\rangle_{\oplus} = 0$.

DIRAC STRUCTURE ASSOCIATED WITH THE EXTERIOR DERIVATIVE (1)

CANONICAL SYMMETRIC FORM

Denote:

$$f = (f_E,\ f_M,\ f_b) \in \Omega^p(N) \times \Omega^q(N) \times \Omega^{n-q}(\partial N)$$
$$e = (e_E,\ e_M,\ e_b) \in \Omega^{n-p}(N) \times \Omega^{n-q}(N) \times \Omega^{n-p}(\partial N)$$

The the canonical symmetric form becomes:

$$\left\langle \left(f_E^1,\ f_M^1,\ f_b^1,\ e_E^1,\ e_M^1,\ e_b^1\right), \left(f_E^2,\ f_M^2,\ f_b^2,\ e_E^2,\ e_M^2,\ e_b^2\right)\right\rangle_{\oplus}$$
$$= \int_N e_E^1 \wedge f_E^2 + e_M^1 \wedge f_M^2 + e_E^2 \wedge f_E^1 + e_M^2 \wedge f_M^1$$
$$+ \int_{\partial N} e_b^1 \wedge f_b^2 + e_b^2 \wedge f_b^1$$

DEFINITION:

Consider $\mathfrak{D} \subset \mathcal{V} \times \mathcal{V}^*$ defined as:

$$\mathfrak{D} = \{ (f_E,\ f_M,\ f_b,\ e_E,\ e_M,\ e_b) \in \mathcal{V} \times \mathcal{V}^* \mid$$
$$\begin{bmatrix} f_E \\ f_M \end{bmatrix} = \begin{bmatrix} 0 & (-1)^{n-k}\mathrm{d} \\ \mathrm{d} & 0 \end{bmatrix} \begin{bmatrix} e_E \\ e_M \end{bmatrix}$$
$$\begin{bmatrix} f_b \\ f_b \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e_E|_{\partial N} \\ e_M|_{\partial N} \end{bmatrix} \}$$

SPACE OF POWER VARIABLES: CONJUGACY

Let N be n–dimensional spatial domain with boundary ∂N.

Consider the *linear space*:

$$\mathcal{V} = \Omega^p(N) \times \Omega^q(N) \times \Omega^{n-q}(\partial N)$$

for $p, \ q$ positive integers, satisfying: $p + q = n + 1$.

Its dual may be identified as follows:

$$\mathcal{V}^* \simeq \Omega^{n-p}(N) \times \Omega^{n-q}(N) \times \Omega^{n-p}(\partial N)$$

using the isomorphism:

$$\begin{array}{ccc} \left(\Omega^k(N)\right)^* & \simeq & \Omega^{n-k}(N) \\ f & \leftrightarrow & \beta_f \in \Omega^{n-k} \end{array}$$

identifying the duality product with:

$$\langle f \mid \alpha \rangle = \int_N \beta_f \wedge \alpha \quad , \qquad \forall \alpha \in \Omega^k(\partial N)$$

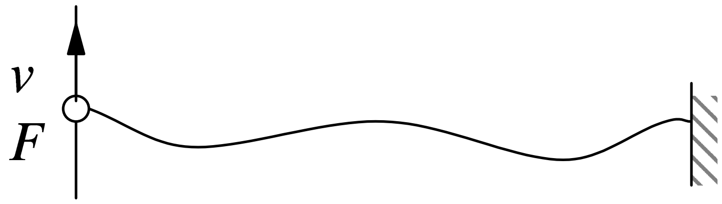
In the sequel: $p = q = k$, hence: $n + 1 = 2k$:

transmission line/ vibrating string: $n = 1, \ k = 1, \ n{-}k = 0$

Maxwell's equations: $n = 3, \ k = 2, \ n{-}k = 1$

MOTIVATIONAL EXAMPLES (2)

VIBRATING STRING:

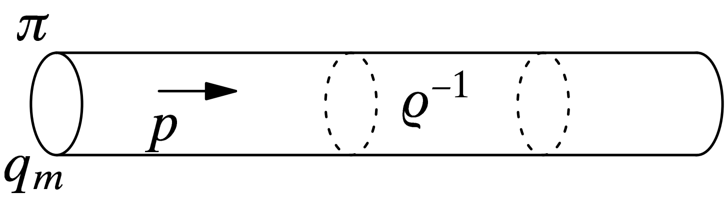


$$\mu\ u_{tt} - T\ u_{zz} = 0$$

$$F = T\ u_z(0)$$

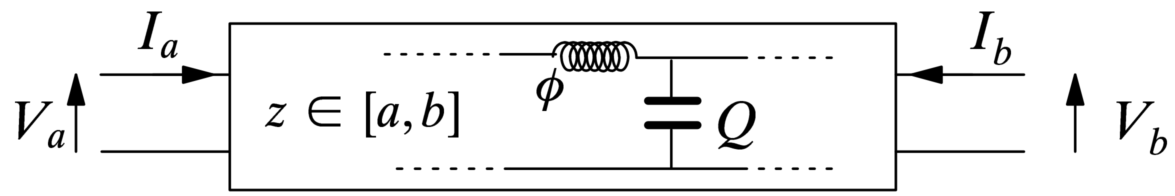
$$v = u_t(0)$$

1–D FLUID DYNAMICS:



MOTIVATIONAL EXAMPLES (1)

TRANSMISSION LINE: Telegrapher's equations



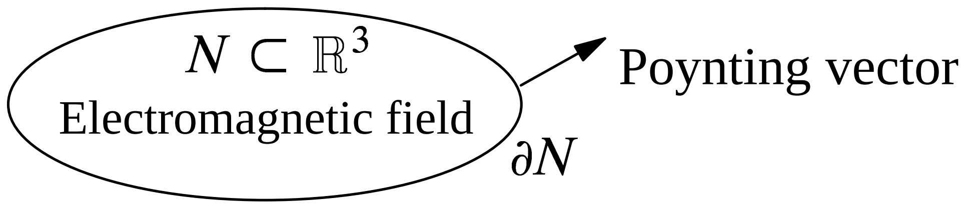
$$\frac{\partial Q}{\partial t}(z,t) = -\frac{\partial}{\partial z} I(z,t) = -\frac{\partial}{\partial z}\frac{\phi(z,t)}{L(z)}$$

$$V_a(t) = V(a,t)$$
$$I_a(t) = -I(a,t)$$

$$\frac{\partial \phi}{\partial t}(z,t) = -\frac{\partial}{\partial z} V(z,t) = -\frac{\partial}{\partial z}\frac{Q(z,t)}{C(z)}$$

$$V_b(t) = V(b,t)$$
$$I_b(t) = -I(b,t)$$

ELECTROMAGNETIC FIELD: Maxwell's equations



$$\frac{\partial D}{\partial t} = \text{curl } H \qquad E = \varepsilon^{-1} D$$

$$\frac{\partial B}{\partial t} = -\text{curl } E \qquad H = \mu^{-1} B$$

Poynting vector
$E \times H$

3. PORT CONTROLLED HAMILTONIAN SYSTEM WITH POWER FLOW THROUGH THE BOUNDARY

Hamiltonian formulation of distributed parameter systems using Poisson structure:

$$\frac{\partial u}{\partial t} = J \ . \ \delta\mathcal{H}[u]$$

$u(t,z)$: state variable, $z \in N$ spatial variables

$\mathcal{H}[u] = \int_N H(z,\ u,\ u_z,\ ..)\ dz$: Hamiltonian functional

$\delta\mathcal{H}[u]$: variational derivative

J : Hamiltonian operator (skew–adjoint, Jacobi identities)

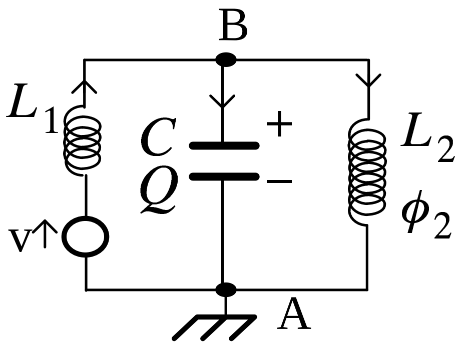
Problems:

1. How to include energy flows through the boundary ? How to describe interaction with other systems ?

2. Skew–symmetry of Poisson structure is based (integration by parts!) on boundary conditions corresponding to zero energy flow.

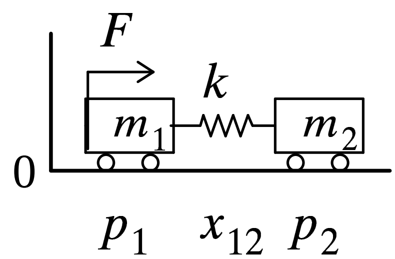
3. Variational derivative a priori includes boundary contributions

EXAMPLE: SCALAR NETWORKS



$$\begin{bmatrix} Q \\ \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \end{bmatrix}$$

$$H(\chi) = \frac{1}{2}\left[\frac{Q^2}{C} + \frac{\phi_1^2}{L_1} + \frac{\phi_2^2}{m_{L2}}\right]$$



$$\begin{bmatrix} x_{12} \\ p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \end{bmatrix}$$

$$H(\chi) = \frac{1}{2}\left[k\ x_{12}^2 + \frac{p_1^2}{m_1} + \frac{p_2^2}{m_2}\right]$$

Third order port–controlled Hamiltonian system:

$$\frac{d}{dt}\begin{bmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial \chi_1} \\ \frac{\partial H}{\partial \chi_2} \\ \frac{\partial H}{\partial \chi_3} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u$$

– Hamiltonian function is *energy function*
– structure matrix given by the *interconnections*

–*input field is not Hamiltonian:* $g \notin \mathrm{im}\, J$

$$y = -\frac{\partial H}{\partial \chi_2} = \begin{cases} -\dfrac{\phi_1}{L_1} & \text{current through the inductor} \\ -\dfrac{p_1}{m_1} & \text{velocity of mass 1} \end{cases}$$

POTENTIAL VERSUS NON–POTENTIAL PORT INTERACTION

PORT CONTROLLED HAMILTONIAN SYSTEM:

– (pseudo) Poisson bracket $\{,\ \}$
– Hamiltonian function $H_0(x)$

port interaction defined by *input vector fields* $g_i(\boldsymbol{x})$:

$$\dot{\boldsymbol{x}} = \{x, H_0\} + \sum_{i=1}^{m} g_i(\boldsymbol{x})\ u_i$$

Output conjugated with inputs by adjoint relation:

$$y_i = < dH_0(x),\ g_i(x) > = g_i^t(x)\ dH_0(x)$$

INPUT–OUTPUT HAMILTONIAN SYSTEMS:

– Poisson bracket $\{,\ \}$
– Hamiltonian function $H_0(x)$

potential interaction defined by

interaction Hamiltonian functions $H_i(\boldsymbol{x})$

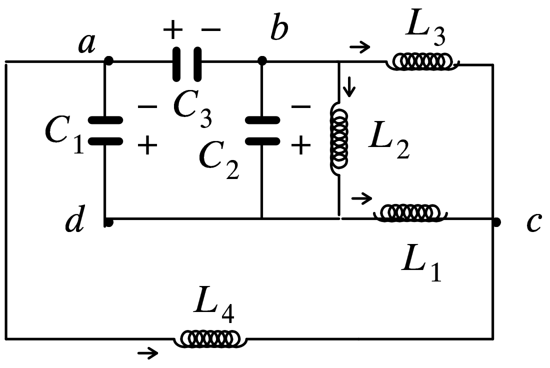
$$\dot{\boldsymbol{x}} = \{x, H_0\} - \sum_{i=1}^{m} \{x, H_i\}\ u_i = \left\{x, \left(H_0 - \sum_{i=1}^{m} u_i\ H_i\right)\right\}$$

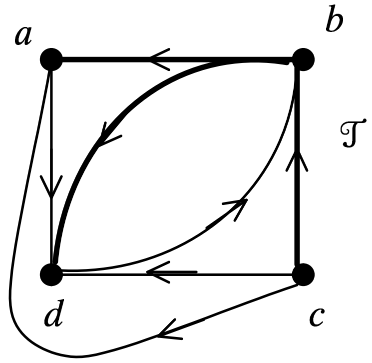
Natural outputs: $\tilde{y}_i = H_i\ (\boldsymbol{x})$

Not all interaction derive from potential

Hamiltonian input/output not suitable for interconnection

EXAMPLE





The maximal tree $\mathcal{T}$ composed of bold edges with partition: $\mathcal{T} = \mathcal{C}_1 \cup \mathcal{L}_2$
where: $\mathcal{C}_1 = \{(b,d),\ (b,a)\}$ contains: C_2 and C_3
and $\mathcal{L}_2 = \{(c,b)\}$ contains: L_3 .

The complementary cotree is $\overline{\mathcal{T}} = \mathcal{L}_1 \cup \mathcal{C}_2$ with partition:
$\mathcal{L}_1 = \{(c,d),\ (d,\ b),\ (c,\ a)\}$ containing: L_1 , L_2 and L_4.
$\mathcal{C}_2 = \{(a,d)\}$ containing: C_1.

Note: order= $|\mathcal{C}_1| + |\mathcal{L}_1| = 5$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_2 \\ \dot{q}_3 \\ \dot{\phi}_1 \\ \dot{\phi}_2 \\ \dot{\phi}_4 \\ \dot{q}_1 \\ \dot{\phi}_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_{C_2} \\ v_{C3} \\ i_{L1} \\ i_{L2} \\ i_{L4} \\ v_{C_1} \\ i_{L3} \end{bmatrix} = 0$$

DEFINITION OF THE DIRAC STRUCTURE ASSOCIATED WITH AN LC CIRCUIT

Let $\mathcal{T}$ be a maximal tree in $\mathcal{G}$, also maximal with respect to the number n_{C_1} of capacitors it contains.

Let $\overline{\mathcal{T}}$ be the complementary cotree of $\mathcal{T}$ maximal with respect to the number n_{L_1} of inductors.

Let $\mathcal{C}_1$ (resp. $\mathcal{C}_2$) be the partial graph of $\mathcal{T}$ (resp. $\overline{\mathcal{T}}$) whose edges correspond to capacitors.

Let $\mathcal{L}_1$ (resp. $\mathcal{L}_2$) be the partial graph of $\overline{\mathcal{T}}$ (resp. $\mathcal{T}$) whose edges correspond to inductors.

$\mathcal{C}_1 \cup \mathcal{L}_2 = \mathcal{T}$ and $\mathcal{L}_1 \cup \mathcal{C}_2 = \overline{\mathcal{T}}$.

Then the Dirac structure is given by:

$$E\left(i_{\mathcal{C}_1}, v_{\mathcal{L}_1},\ i_{\mathcal{C}_2},\ v_{\mathcal{L}_2}\right)^t + F\left(v_{\mathcal{C}_1}, i_{\mathcal{L}_1},\ i_{v\mathcal{C}_2},\ i_{\mathcal{L}_2}\right)^t = 0$$

defined by the cycle and cocycle matrices generated by the partition of the port connection graph:

$$E = \begin{bmatrix} I_{n_{C_1}} & 0 & -B_{21}^t & 0 \\ 0 & I_{n_{L_1}} & 0 & B_{12} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad F = \begin{bmatrix} 0 & -B_{11}^t & 0 & 0 \\ B_{11} & 0 & 0 & 0 \\ -B_{21} & 0 & -I_{n_{C_2}} & 0 \\ 0 & B_{12}^t & 0 & -I_{n_{L_2}} \end{bmatrix}.$$

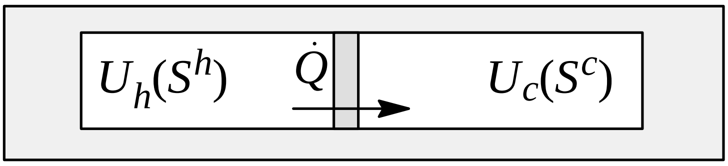
The associated Dirac structure is then:

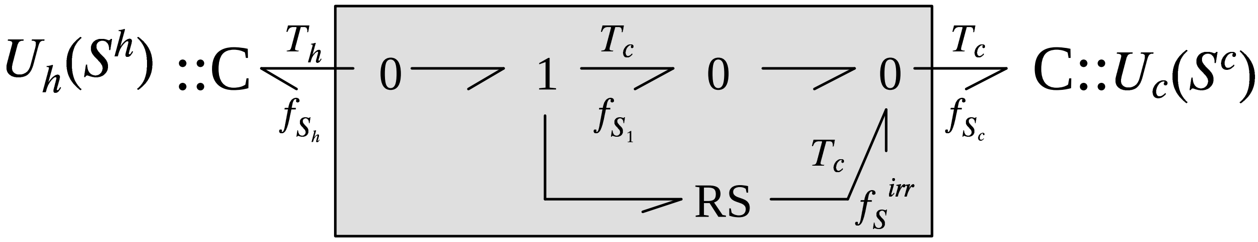
$$E\underbrace{\left(i_{\mathcal{C}_1}, u_{\mathcal{L}_1}, i_{\mathcal{C}_2}, u_{\mathcal{L}_2}\right)^t}_{\text{rate variables } \dot{x}} + F\underbrace{\left(v_{\mathcal{C}_1}, i_{\mathcal{L}_1}, v_{\mathcal{C}_2}, i_{\mathcal{L}_2}\right)^t}_{\text{coenergy variables } \frac{\partial H}{\partial x}} = 0.$$

2 GASES IN THERMAL INTERACTION

[Maschke and Chantre 1994, Maschke 1998]

Consider 2 gases, denoted by *c* and *h*:
–in ***thermal interaction*** through a wall obeying Fourier's conduction law
– in a rigid, closed reservoir thermally isolated from the environment.





State variables: entropies $\boldsymbol{S} = \left(S^h,\ S^c\right)^t \in \mathbb{R}^2$

Energy function: sum of the internal energies:

$$H(S^h, S^c) = U_h(S^h) + U_c(S^c)$$

coenergy variables: $dH(\boldsymbol{S}) = \left(T^h,\ T^c\right)^t$

Hamiltonian system: $\frac{d\boldsymbol{S}}{dt} = J(\boldsymbol{T})\ dH(\boldsymbol{S})$

Poisson structure associated with conduction:

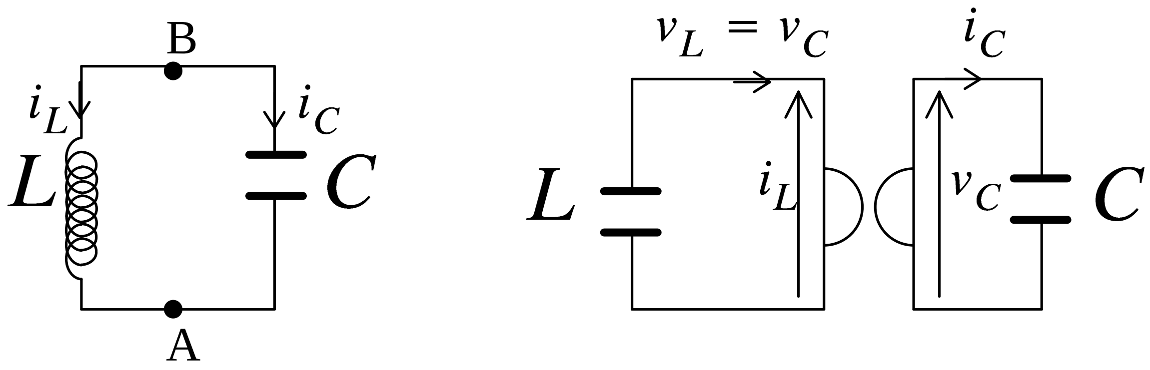
$$J(\boldsymbol{T}) = R\left(\frac{1}{T^c} - \frac{1}{T^h}\right)\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

interconnection = Fourier's heat conduction law.

SYMPLECTIC INTERCONNECTION

[Breedveld 1982, Maschke 1991]

Consider an LC oscillator:



State variables: energy variables
charge and magnetic flux $x = (Q,\ \phi)^t \in \mathbb{R}^2$

Energy function: sum of electric and magnetic energies:

$$H(Q,\ \phi) = H_{el}(Q) + H_{mg}(\phi)$$

coenergy variables: $dH(x) = (v_C,\ i_L)^t$

Hamiltonian system: $\dfrac{dx}{dt} = J^s\ dH(x)$

Poisson structure associated to electromagnetic coupling:

$$J^s = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

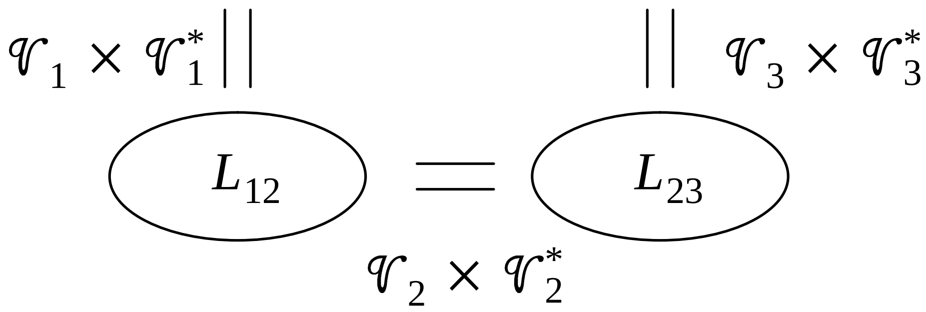
interconnection.

Elementary electromagnetic coupling: *gyrator*

POWER CONTINUOUS INTERCONNECTION

INTERCONNECTION OF DIRAC STRUCTURES

[van der Schaft 2000]



Consider 2 Dirac structures L_{12} and L_{23} defined on the product spaces $\mathcal{V}_1 \times \mathcal{V}_1^* \times \mathcal{V}_2 \times \mathcal{V}_2^*$ and $\mathcal{V}_2 \times \mathcal{V}_2^* \times \mathcal{V}_3 \times \mathcal{V}_3^*$ with respect to the bilinear forms:

$$\left\langle \left(v_i^a, w_i^a, v_j^a, w_j^a\right), \ \left(v_i^b, w_i^b, v_j^b, w_j^b\right)\right\rangle_{ij}$$

$$= \left\langle w_1^a, v_1^b \right\rangle + \left\langle w_1^b, v_1^a \right\rangle - \left\langle w_2^a, v_2^b \right\rangle - \left\langle w_2^b, v_2^a \right\rangle$$

then

$$L_{13} = \left\{ \left(v_1, w_1, v_3, w_3\right) \in \mathcal{V}_1 \times \mathcal{V}_1^* \times \mathcal{V}_3 \times \mathcal{V}_3^* \right.$$

$$\left. \exists \left(v_2, w_2\right) \in \mathcal{V}_2 \times \mathcal{V}_2^* \ | \left(v_1, w_1, v_2, w_2\right) \in L_{12} \ , \ \left(v_1, w_1, v_2, w_2\right) \in L_{12} \right\}$$

POWER CONTINUOUS INTERCONNECTION OF PORT CONTROLLED HAMILTONIAN SYSTEMS

[Dalsmo and van der Schaft 1999, van der Schaft 2000]

Consider 2 port controlled Hamiltonian systems (indexed by 1 and 2), the power continuous interconnection of Dirac structures through a third Dirac structure define a port controlled Hamiltonian system.

PORT CONTROLLED HAMILTONIAN SYSTEM

[van der Schaft and Maschke 1995]

A port controlled Hamiltonian system with constant interconnection structure is defined by:

– a state space: a vector space $\mathcal{X}$

– a space of port variables: vector space $W \oplus W^*$

– a Dirac structure : $L \subset \mathcal{X} \oplus W \oplus \mathcal{X}^* \oplus W^*$

– a Hamiltonian function $H : \mathcal{X} \rightarrow \mathbb{R}$,

and the dynamical system:

$$\boxed{\left(-\dot{x}\ ,\ w_f\ , dH,\ w_e\right)\ \in L}$$

The Hamiltonian function is conserved:

$$\frac{dH}{dt} = \langle dH,\ \dot{x} \rangle = \left\langle w_f\ ,\ w_e \right\rangle.$$

REPRESENTATIONS OF DIRAC STRUCTURES (2)

Effort constraint representation

[Dalsmo and van der Schaft 1999]

A Dirac structure L on the n–dimensional real vector space $\mathcal{V}$ is defined by: – a subspace P_1 of $\mathcal{V} \oplus \mathcal{V}^*$ and

– a *skew–symmetric map* J from P_1 to P_1^*.

Scattering representation [van der Schaft 2000]

A Dirac structure L on the n–dimensional real vector space $\mathcal{V}$ is defined by *an isometry* $\mho$ from S_+ to S_- the subspaces of scattering variables associated with the singular values 1 and -1 of the canonical symmetric form $<,>_+$.

REPRESENTATIONS OF DIRAC STRUCTURES (1)

Image representation [Courant 1990]

A Dirac structure L on the n–dimensional real vector space $\mathcal{V}$ is defined by :

$$\boxed{L = \mathrm{Im}\ a \oplus \mathrm{Im}\ b}$$

where a and b are 2 linear maps $a : \mathbb{R}^n \to \mathcal{V}$ and $b : \mathbb{R}^n \to \mathcal{V}^*$:

$$a^* b + b^* a = 0 \quad \text{and} \quad \ker a \cap \ker b = \{0\}.$$

Kernel representation [van der Schaft and Maschke 1995]

A Dirac structure L on the n–dimensional real vector space $\mathcal{V}$ is defined by :

$$\boxed{L = \ker(F + E)}$$

where E and F are linear maps $F : \mathcal{V} \to \mathbb{R}^n$ and $E : \mathcal{V}^* \to \mathbb{R}^n$:

$$E F^* + F E^* = 0 \quad \text{and} \quad \mathrm{rank}\ [E;F] = n.$$

Hybrid input–output representation

[Bloch and Crouch 1999]

A Dirac structure L on the n–dimensional real vector space $\mathcal{V}$ is defined by: – a partition $\mathcal{V}_1 \oplus \mathcal{V}_2 = \mathcal{V}$ and

– a *skew–symmetric map* from $\mathcal{V}_1^* \oplus \mathcal{V}_2$ to $\mathcal{V}_1 \oplus \mathcal{V}_2^*$.

DIRAC STRUCTURE ON A VECTOR SPACE

[Courant 1990] [Dorfman 1993]

CANONICAL SYMMETRIC FORM

For any two pairs (v_1, w_1) and (v_2, w_2) in $\mathcal{V} \oplus \mathcal{V}^*$ define:

$$\langle (v_1, w_1), (v_2, w_2) \rangle_{\oplus} = \langle w_1, v_2 \rangle + \langle w_2, v_1 \rangle$$

where $\langle v_i, w_i \rangle$ is the duality product of a pair $(v_i, w_i) \in \mathcal{V} \times \mathcal{V}^*$.

DIRAC STRUCTURE ON A VECTOR SPACE

A Dirac structure on a vector space $\mathcal{V}$ is a subspace $L \subset \mathcal{V} \oplus \mathcal{V}^*$ which is ***maximally isotropic*** under $<,>_+$ that is a subspace $L \subset \mathcal{V} \oplus \mathcal{V}^*$ such that:

i) $\dim L = n$ and:

ii) $\forall\ (\alpha_1, \alpha_2) \in L \times L,\ \langle \alpha_1,\ \alpha_2 \rangle_{\oplus} = 0$

Condition *ii)* is equivalent with: [van der Schaft and Maschke 1995]

$$\forall\ (v, w) \in L, \quad <w, v> = 0$$

Dirac structures are a generalisation of both:

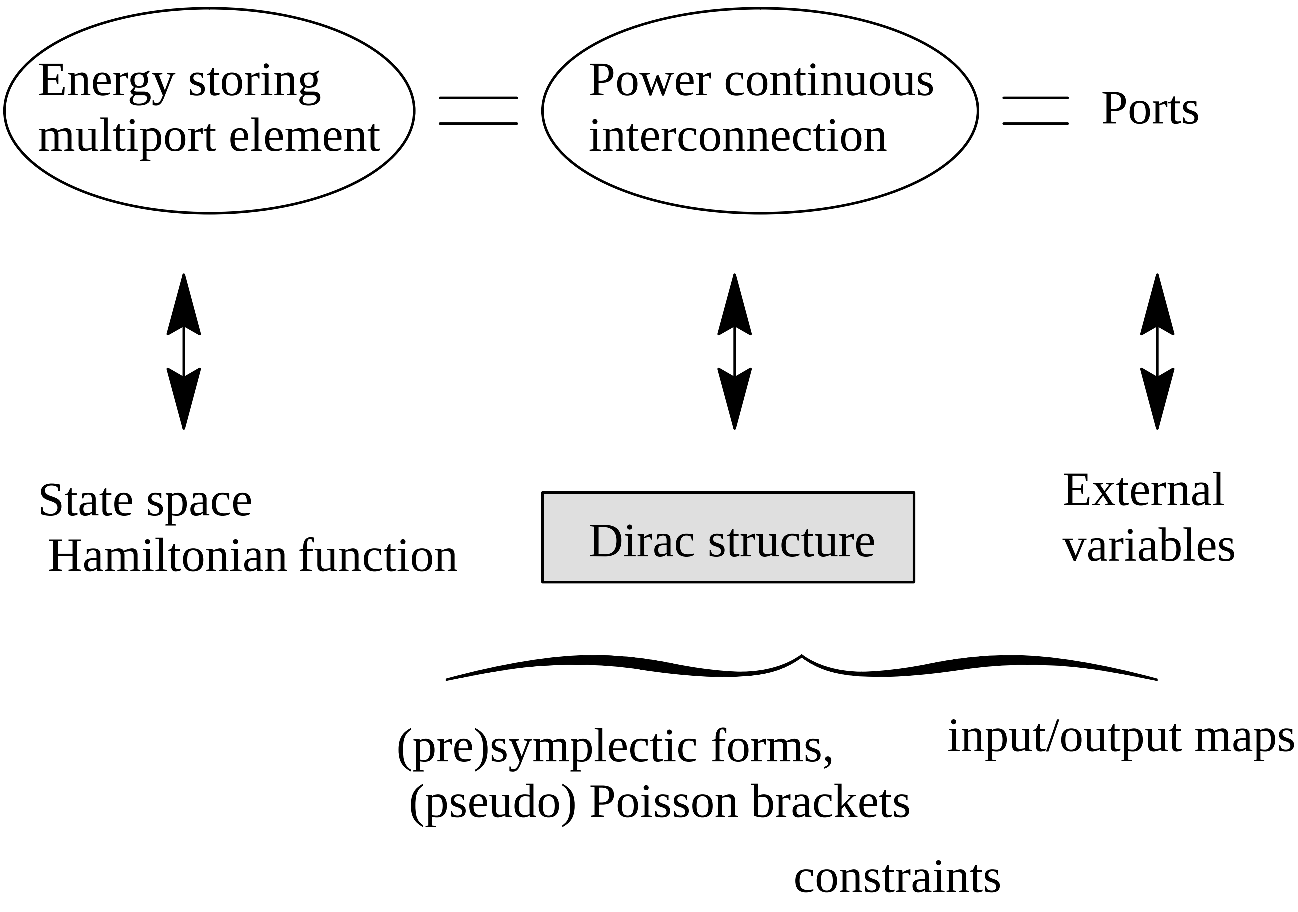
– presymplectic vector spaces

– Poisson vector spaces

and may be associated with Lagrangian submanifold.

2. DIRAC STRUCTURES AND FINITE DIMENSIONAL PORT CONTROLLED HAMILTONIAN SYSTEMS

Geometric and network structures of physical systems' dynamics



– interconnection structure

– port interaction

1. INTRODUCTION

Motivation: network perspective on
physically–based system theory

– tearing

– interconnection

for modeling, simulation and control design.

[Kirchhoff, ..., Kron, .. Paynter, .. Oster, ..., Davies, ..]

Synthesis of (differential–)**geometric** structures
and **network** structures

[Kron, .. Brayton–Moser, Smale, .. Oster, ..., ..]

Based on the **bond graph** formalism

[Paynter 61, Breedveld 84]

For **lumped parameter systems**:

– network interconnection = (non closed) Dirac structures

– port interaction = non potential interaction

implicit ***port controlled Hamiltonian systems***

[van der Schaft and Maschke 95], [van der Schaft 2000], [Maschke 98]

Here we consider **distributed parameter systems**:
– definition of *network* variables
– port interaction through boundary

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for Nonlinear Control,
Princeton , USA, 16–18 March 2000

Port controlled Hamiltonian representation of distributed parameter systems

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