Implicit port-controlled Hamiltonian systems*

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1 Introduction

In the companion paper ”Port-controlled Hamiltonian systems: towards a theory for control and design of nonlinear physical systems” [22] the importance of a theoretical framework for modeling has been stressed, both for the analysis and the control of physical engineering systems. It has been argued that the network modeling of (lumped-parameter) physical systems, for instance using the bond-graph formalism [19, 2], directly leads to a generalized Hamiltonian description, specified by a power-conserving interconnection, a Hamiltonian representing the total energy in the energy-storing elements, and termination of some of the system ports by resistive elements. The resulting class of systems, called port-controlled Hamiltonian systems (with dissipation) includes mechanical systems (multi-body systems, systems with kinematic constraints), electrical circuits, as well as electromechanical systems. Furthermore, it has been indicated that the power-conserving interconnection admits a geometrical formulation as a (generalized) Poisson structure. The usefulness of this geometric framework for analysis and control has been demonstrated on the problem of stabilization of port-controlled Hamiltonian systems, providing a geometric basis for passivity-based control.
A limitation of the framework exposed in the companion paper [22] is that it only deals with physical systems *without* constraints, or systems obtained *after* the elimination of the constraints. It is well-known that from a general modeling point of view physical systems are often described, at least in first instance, as a mixed set of differential and algebraic equations (DAE's). This stems from the fact that by interconnecting systems one usually introduces algebraic constraints between the state variables of the individual systems. Since especially in the nonlinear case the elimination of the algebraic constraints is often fraught with difficulties one would like to extend the framework of port-controlled Hamiltonian systems as discussed in [22] to the setting of DAE's. This will be done in Section 3 by the introduction of *Dirac structures*, which are geometrical objects generalizing Poisson structures. Implicit port-controlled Hamiltonian systems (“Hamiltonian DAE's”) are then defined with respect to a Dirac structure, a Hamiltonian function representing the energy at the energy-storing elements, and a resistive structure.

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Furthermore, in Section 4 we give a geometric treatment of *scattering*, enabling us to directly relate port-controlled Hamiltonian systems to nonlinear *inner* (or *all-pass*) systems, or systems having L_2-gain less than or equal to one. It will be shown that in the scattering representation Dirac structures are given by orthonormal maps. These results open up the way to robust control of physical systems using tools from nonlinear H_∞-control.

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2 Port-controlled Hamiltonian systems

As discussed in the companion paper [22] a major generalization of standard Hamiltonian systems with external forces is to consider systems which are described in local coordinates $x = (x_1, \dots, x_n)$ for an n-dimensional state space manifold $\mathcal{X}$ as

$$\begin{array}{rcll} \dot{x} & = & J(x)\frac{\partial H}{\partial x}(x) + g(x)u, & x \in \mathcal{X}, u \in \mathbb{R}^m \\ y & = & g^T(x)\frac{\partial H}{\partial x}(x), & y \in \mathbb{R}^m \end{array} \tag{1}$$

where $J(x)$ is an $n \times n$ matrix with entries depending smoothly on x, which is assumed to be *skew-symmetric*

$$J(x) = -J^T(x), \tag{2}$$

and H is the Hamiltonian function, representing the total energy stored in the system. Because of the skew-symmetry of J we easily deduce the energy-balance

$$\frac{dH}{dt}(x(t)) = u^T(t)y(t), \tag{3}$$

showing that (1) is passive (in fact, lossless) if $H \geq 0$ (or, *bounded from below*), cf. [28]. The system (1) with J satisfying (2) is called a *port-controlled Hamiltonian (PCH) system* with *structure matrix* $J(x)$ and *Hamiltonian* H.

Port-controlled Hamiltonian systems (1) not only can be seen as a generalization of the Hamiltonian equations of motion, but they directly arise from a *network modeling* of (complex) physical systems without dissipative elements, see e.g. our papers [11, 12, 8, 13, 23, 14, 27, 20, 22]. Indeed, the pair $(J(x), g(x))$, $x \in \mathcal{X}$, captures the *interconnection structure* of the system, with $g(x)$ modeling in particular the *ports* of the system. A very important property which may be directly inferred from the interconnection structure is the existence of dynamical invariants *independent* of the Hamiltonian H, called *Casimir functions*, cf. [22]. Consider the set of p.d.e.'s

$$\frac{\partial^T C}{\partial x}(x)J(x) = 0, \qquad x \in \mathcal{X}, \tag{4}$$

in the unknown (smooth) function $C : \mathcal{X} \to \mathbb{R}$. If (4) has a solution C then it follows that the time-derivative of C along the port-controlled Hamiltonian system (1) satisfies

$$\begin{array}{rcl} \frac{dC}{dt} & = & \frac{\partial^T C}{\partial x}(x)J(x)\frac{\partial H}{\partial x}(x) + \frac{\partial^T C}{\partial x}(x)g(x)u \\ & = & \frac{\partial^T C}{\partial x}(x)g(x)u \end{array} \tag{5}$$

Hence, for the input $u=0$, or for *arbitrary* input functions if additionally $\frac{\partial^T C}{\partial x}(x)g(x)=0$, the function $C(x)$ *remains constant* along the trajectories of the port-controlled Hamiltonian system, *irrespective* of the precise form of the Hamiltonian H. A function $C:\mathcal{X}\to\mathbb{R}$ satisfying (4) is called a *Casimir function* (of the structure matrix $J(x)$). The existence of Casimir functions has immediate consequences for stability analysis of (1) for $u=0$. Indeed, if $C_1,\cdots,C_r$ are Casimirs, then by (4) not only $\frac{dH}{dt}=0$ for $u=0$, but

$$\frac{d}{dt}\left(H+H_a(C_1,\cdots,C_r)\right)(x(t))=0 \tag{6}$$

for *any* function $H_a:\mathbb{R}^r\to\mathbb{R}$. Hence, even if H is not positive definite at an equilibrium $x^*\in\mathcal{X}$, then $H+H_a(C_1,\cdots,C_r)$ may be positive definite at x^* by a proper choice of H_a, and thus may serve as a Lyapunov function. This method for stability analysis is called the *Energy-Casimir method*, see e.g. [7]. In the companion paper [22] it has been discussed (based on [9, 27, 10, 18, 21]) how this framework can be exploited for *stabilization* of PCH systems, by feedback interconnection of the (plant) PCH system with a (controller) PCH system, leading to a *closed-loop* PCH system with a closed-loop interconnection structure matrix J inducing Casimir functions for the closed-loop system. Also, this provides a link with *passivity-based control*, see e.g. [17], as well as [6] for stabilization of mechanical systems with nonholonomic constraints within the PCH framework.
Energy-dissipation is included in the framework of port-controlled Hamiltonian systems (1) by terminating some of the ports by resistive elements, leading to *port-controlled Hamiltonian systems with dissipation* (PCHD systems), cf. [21, 22].

3 Port-controlled Hamiltonian systems with algebraic constraints

From a general modeling point of view physical systems are, at least in first instance, often described by DAE's, that is, a mixed set of differential and algebraic equations. This stems from the fact that in many modeling approaches the system under consideration is naturally regarded as obtained from interconnecting simpler sub-systems. These interconnections in general, give rise to algebraic constraints between the state space variables of the sub-systems; thus leading to implicit systems. While in the linear case one may argue that it is often relatively straightforward to eliminate the algebraic constraints, and thus to reduce the system to an *explicit* form without constraints, in the nonlinear case such a conversion from implicit to explicit form is usually fraught with difficulties. Indeed, if the algebraic constraints are nonlinear then they need not be analytically solvable (locally or globally). More importantly perhaps, even if they are analytically solvable, then often one would prefer *not* to eliminate the algebraic constraints, because of the complicated and physically not easily interpretable expressions for the reduced system which may arise. Therefore it is important to extend the framework of port-controlled Hamiltonian systems, as sketched in the companion paper [22] and the previous section, to the context of systems with algebraic constraints or *implicit systems.*

3.1 Power-conserving interconnections

In order to give the definition of an implicit port-controlled Hamiltonian system we first consider the notion of a *Dirac structure,* formalizing the concept of a power-conserving interconnection, and generalizing the notion of a structure matrix $J(x)$ as encountered before. Let $\mathcal{F}$ be an ℓ-dimensional linear space, and denote its dual (the space of linear functions on $\mathcal{F}$) by $\mathcal{F}^*$. The product space $\mathcal{F}\times\mathcal{F}^*$ is considered to be the space of power variables, with *power* intrinsically defined by

$$P=< f^*|f>, \quad (f,f^*)\in\mathcal{F}\times\mathcal{F}^*, \tag{7}$$

where $< f^*| f >$ denotes the duality product, that is, the linear function $f^* \in \mathcal{F}^*$ acting on $f \in \mathcal{F}$. Often we call $\mathcal{F}$ the space of *flows* f, and $\mathcal{F}^*$ the space of *efforts* e, with the power of an element $(f, e) \in \mathcal{F} \times \mathcal{F}^*$ denoted as $< e| f >$.

Remark 3.1 If $\mathcal{F}$ is endowed with an *inner product* structure $<, >$, then $\mathcal{F}^*$ can be naturally *identified* with $\mathcal{F}$ in such a way that $< e| f >=< e, f >, \ f \in \mathcal{F}, \ e \in \mathcal{F}^* \simeq \mathcal{F}$.

Example 3.2 Let $\mathcal{F}$ be the space of generalized *velocities*, and $\mathcal{F}^*$ be the space of generalized *forces*, then $< e| f >$ is mechanical power. Similarly, let $\mathcal{F}$ be the space of *currents*, and $\mathcal{F}^*$ be the space of *voltages*, then $< e| f >$ is electrical power.

There exists on $\mathcal{F} \times \mathcal{F}^*$ a canonically defined symmetric bilinear form

$$< (f_1, e_1), (f_2, e_2) >_{\mathcal{F} \times \mathcal{F}^*} := < e_1| f_2 > + < e_2| f_1 > \tag{8}$$

for $f_i \in \mathcal{F}, \ e_i \in \mathcal{F}^*, i = 1, 2$. Now consider a linear subspace $S \subset \mathcal{F} \times \mathcal{F}^*$, and its orthogonal complement with respect to the bilinear form $<, >_{\mathcal{F} \times \mathcal{F}^*}$ on $\mathcal{F} \times \mathcal{F}^*$, denoted as $S^\perp \subset \mathcal{F} \times \mathcal{F}^*$. Clearly, if S has dimension d, then the subspace $S^\perp$ has dimension $2\ell - d$. (Since dim $(\mathcal{F} \times \mathcal{F}^*) = 2\ell$, and $<, >_{\mathcal{F} \times \mathcal{F}^*}$ is a non-degenerate form.)

Definition 3.3 [3, 5, 4] A constant Dirac structure on $\mathcal{F}$ is a linear subspace $\mathcal{D} \subset \mathcal{F} \times \mathcal{F}^*$ such that

$$\mathcal{D} = \mathcal{D}^\perp \tag{9}$$

It immediately follows that the dimension of any Dirac structure $\mathcal{D}$ on an ℓ-dimensional linear space is equal to ℓ. Furthermore, let $(f, e) \in \mathcal{D} = \mathcal{D}^\perp$. Then by (8)

$$0 = < (f, e), (f, e) >_{\mathcal{F} \times \mathcal{F}^*} = 2 < e| f > . \tag{10}$$

Thus for all $(f, e) \in \mathcal{D}$ we obtain $< e| f >= 0$; and hence any Dirac structure $\mathcal{D}$ on $\mathcal{F}$ defines a power-conserving relation between the power variables $(f, e) \in \mathcal{F} \times \mathcal{F}^*$.

Remark 3.4 The property dim $\mathcal{D} =$ dim $\mathcal{F}$ is intimately related to the usually expressed statement that a physical interconnection can *not* determine at the same time both the flow and effort (e.g. current *and* voltage, or velocity *and* force).

Constant Dirac structures admit different *matrix representations.* Here we just list a number of them, without giving proofs and algorithms to convert one representation into another, see e.g. [4].
Let $\mathcal{D} \subset \mathcal{F} \times \mathcal{F}^*$, with dim $\mathcal{F} = \ell$, be a constant Dirac structure. Then $\mathcal{D}$ can be represented as

1. (*Kernel and Image representation,* [4, 25]).

   $$\mathcal{D} = \{(f, e) \in \mathcal{F} \times \mathcal{F}^* | Ff + Ee = 0\} \tag{11}$$

   for $\ell \times \ell$ matrices F and E satisfying

   $$\begin{aligned} &(i) \quad EF^T + FE^T = 0 \\ &(ii) \quad \text{rank } [F \vdots E] = \ell \end{aligned} \tag{12}$$

   Equivalently,

   $$\mathcal{D} = \{(f, e) \in \mathcal{F} \times \mathcal{F}^* | f = E^T\lambda, \ e = F^T\lambda, \ \lambda \in \mathbb{R}^\ell\} \tag{13}$$

2. (*Constrained input-output representation*, [4]).

$$\mathcal{D} = \{(f, e) \in \mathcal{F} \times \mathcal{F}^* | \, f = -Je + G\lambda, \; G^T e = 0\} \tag{14}$$

for an $\ell \times \ell$ skew-symmetric matrix J, and a matrix G such that $\mathrm{Im} G = \{f | (f, 0) \in \mathcal{D}\}$. Furthermore, $\mathrm{Ker} J = \{e | (0, e) \in \mathcal{D}\}$.

3. (*Hybrid input-output representation*, [1]).
Let $\mathcal{D}$ be given as in (11). Suppose rank $F = \ell^1 (\leq \ell)$. Select ℓ^1 independent columns of F, and group them into a matrix F^1. Write (possibly after permutations) $F = [F^1 \vdots F^2]$ and, correspondingly $E = [E^1 \vdots E^2]$, $f = \begin{bmatrix} f^1 \\ f^2 \end{bmatrix}, e = \begin{bmatrix} e^1 \\ e^2 \end{bmatrix}$. Then the matrix $[F^1 \vdots E^2]$ can be shown to be invertible, and

$$\mathcal{D} = \left\{ \begin{pmatrix} f^1 \\ f^2 \end{pmatrix}, \begin{pmatrix} e^1 \\ e^2 \end{pmatrix} \middle| \begin{pmatrix} f^1 \\ e^2 \end{pmatrix} = -J \begin{pmatrix} e^1 \\ f^2 \end{pmatrix} \right\} \tag{15}$$

with $J := \left[F^1 \vdots E^2 \right]^{-1} \left[F^2 \vdots E^1 \right]$ skew-symmetric.

4. (*Canonical coordinate representation*, [3]).
There exist linear coordinates (q, p, r, s) for $\mathcal{F}$ such that in these coordinates and dual coordinates for $\mathcal{F}^*$, $(f, e) = (f_q, f_p, f_r, f_s, e_q, e_p, e_r, e_s) \in \mathcal{D}$ if and only if

$$\left\{ \begin{array}{lllllll} f_q & = & e_p, & f_p & = & -e_q \\ \\ f_r & = & 0, & e_s & = & 0 \end{array} \right. \tag{16}$$

Example 3.5 Kirchhoff's laws are a special case of (11). By taking $\mathcal{F}$ the space of currents and $\mathcal{F}^*$ the space of voltages, Kirchhoff's current laws determine a subspace $\mathcal{V}$ of $\mathcal{F}$, while Kirchhoff's voltage laws determine the orthogonal subspace $\mathcal{V}^{orth}$ of $\mathcal{F}^*$. Hence, the Dirac structure determined by Kirchhoff's laws is given as $\mathcal{V} \times \mathcal{V}^{orth} \subset \mathcal{F} \times \mathcal{F}^*$, with kernel representation of the form

$$\mathcal{D} = \{(f, e) \in \mathcal{F} \times \mathcal{F}^* | Ff = 0, Ee = 0\}, \tag{17}$$

for suitable matrices F and E (consisting only of elements $+1$, -1 and 0), such that Ker $F = \mathcal{V}$ and Ker $E = \mathcal{V}^{orth}$. In this case the defining property $\mathcal{D} = \mathcal{D}^\perp$ of the Dirac structure amounts to Tellegen's theorem.

Example 3.6 Any skew-symmetric map $J : \mathcal{F}^* \to \mathcal{F}$ defines the Dirac structure

$$\mathcal{D} = \{(f, e) \in \mathcal{F} \times \mathcal{F}^* | \, f = -Je\}, \tag{18}$$

as a special case of (14). Furthermore, any interconnection structure (J, g) with J skew-symmetric defines a Dirac structure given in hybrid input-output representation as

$$\begin{bmatrix} f_S \\ e_P \end{bmatrix} = \begin{bmatrix} -J & -g \\ g^T & 0 \end{bmatrix} \begin{bmatrix} e_S \\ f_P \end{bmatrix} \tag{19}$$

Given a Dirac structure $\mathcal{D}$ on $\mathcal{F}$, the following subspaces of $\mathcal{F}$, respectively $\mathcal{F}^*$, will shown to be of importance in the next section

$$
\begin{aligned}
G_1 &:= \{f \in \mathcal{F} \mid \exists e \in \mathcal{F}^* \text{ s.t. } (f, e) \in \mathcal{D}\} \\
P_1 &:= \{e \in \mathcal{F}^* \mid \exists f \in \mathcal{F} \text{ s.t. } (f, e) \in \mathcal{D}\}
\end{aligned}
\tag{20}
$$

The subspace G_1 expresses the set of admissible flows, and P_1 the set of admissible efforts. In the image representation (13) they are given as

$$
G_1 = \operatorname{Im} E^T, \qquad P_1 = \operatorname{Im} F^T. \tag{21}
$$

3.2 Implicit port-controlled Hamiltonian systems

From a network modeling perspective, see e.g. [19, 2], a (lumped-parameter) physical system is directly described by a set of (possibly multi-dimensional) *energy-storing* elements, a set of *energy-dissipating* or *resistive* elements, and a set of *ports* (by which interaction with the environment can take place), interconnected to each other by a *power-conserving interconnection*, see Figure 1.

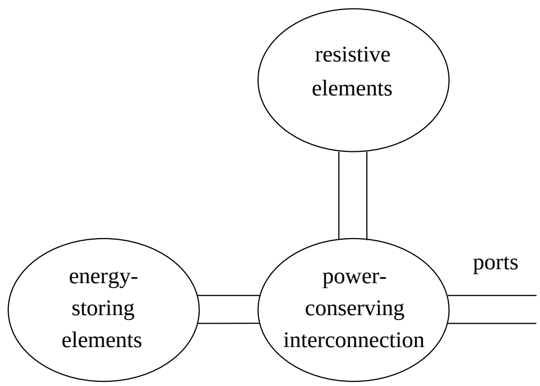

Figure 1: Implicit port-controlled Hamiltonian system with dissipation

Associated with the energy-storing elements are energy-variables $x_1, \cdots, x_n$, being coordinates for some n-dimensional state space manifold $\mathcal{X}$, and a total energy $H : \mathcal{X} \to \mathbb{R}$. The power-conserving interconnection also includes power-conserving elements like (in the electrical domain) transformers, gyrators, or (in the mechanical domain) transformers, kinematic pairs and kinematic constraints. In first instance (see later on for the non-constant case) the power-conserving interconnection will be formalized by a constant Dirac structure on a finite-dimensional linear space $\mathcal{F} := \mathcal{F}_S \times \mathcal{F}_R \times \mathcal{F}_P$, with $\mathcal{F}_S$ denoting the space of flows f_S connected to the energy-storing elements, $\mathcal{F}_R$ denoting the space of flows f_R connected to the dissipative (resistive) elements, and $\mathcal{F}_P$ the space of external flows f_P which can be connected to the environment. Dually, we write $\mathcal{F}^* = \mathcal{F}_S^* \times \mathcal{F}_R^* \times \mathcal{F}_P^*$, with $e_S \in \mathcal{F}_S^*$ the efforts connected to the energy-storing elements, $e_R \in \mathcal{F}_R^*$ the efforts connected to the resistive elements, and $e_P \in \mathcal{F}_P^*$ the efforts to be connected to the environment of the system.
In kernel representation, the Dirac structure on $\mathcal{F} = \mathcal{F}_S \times \mathcal{F}_R \times \mathcal{F}_P$ is given as

$$
\begin{aligned}
\mathcal{D} = \{&(f_S, f_R, f_P, e_S, e_R, e_P) \mid \\
&F_S f_S + E_S e_S + F_R f_R + E_R e_R + F_P f_P + E_P e_P = 0\}
\end{aligned}
\tag{22}
$$

for certain matrices F_S, E_S, F_R, E_R, F_P, E_P satisfying

$$\begin{array}{ll} (i) & E_S F_S^T + F_S E_S^T + E_R F_R^T + F_R E_R^T + E_P F_P^T + F_P E_P^T = 0 \\ (ii) & \text{rank} \left[F_S \vdots F_R \vdots F_P \vdots E_S \vdots E_R \vdots E_P \right] = \dim \mathcal{F} \end{array} \tag{23}$$

The flow variables of the energy-storing elements are given as $\dot{x}(t) = \frac{dx}{dt}(t), t \in \mathbb{R}$, and the effort variables of the energy-storing elements as $\frac{\partial H}{\partial x}(x(t))$ (implying that $< \frac{\partial H}{\partial x}(x(t)) | \dot{x}(t) >= \frac{dH}{dt}(x(t))$ is the increase in energy). In order to have a consistent sign convention for energy flow we put

$$\begin{array}{rcl} f_S & = & -\dot{x} \\ e_S & = & \frac{\partial H}{\partial x}(x) \end{array} \tag{24}$$

Restricting to *linear* resistive elements, the flow and effort variables connected to the resistive elements are related as

$$f_R = -S e_R \tag{25}$$

for some matrix $S = S^T \geq 0$. Substitution of (24) and (25) into (22) yields

$$-F_S \dot{x}(t) + E_S \frac{\partial H}{\partial x}(x(t)) - F_R S e_R + E_R e_R + F_P f_P + E_P e_P = 0 \tag{26}$$

with F_S, E_S, F_R, E_R, F_P, E_P satisfying (23). We call (26) an *implicit port-controlled Hamiltonian system with dissipation,* defined with respect to the constant Dirac structure $\mathcal{D}$, the Hamiltonian H, and the resistive structure S..

Remark 3.7 Under certain full rank conditions the *elimination* of the algebraic constraints (31) for an implicit PCHD system (26) can be shown to result in an *explicit* PCHD system on a lower-dimensional state space, see e.g. [21].

Actually, for many purposes this definition of an implicit PCHD system is not general enough, since often the Dirac structure is not constant, but *modulated* by the state variables x. In this case the matrices F_S, E_S, F_R, E_R, F_P, E_P depend (smoothly) on x, leading to the implicit PCHD system

$$\begin{array}{l} -F_S(x(t))\dot{x}(t) + E_S(x(t)) \frac{\partial H}{\partial x}(x(t)) - F_R(x(t)) R e_R(t) \\ + E_R(x(t)) e_R(t) + F_P(x(t)) f_P(t) + E_P(x(t)) e_P(t) = 0, \quad t \in \mathbb{R} \end{array} \tag{27}$$

with

$$\begin{array}{l} E_S(x) F_S^T(x) + F_S(x) E_S^T(x) + E_R(x) F_R^T(x) + F_R(x) E_R^T(x) \\ + \; E_P(x) F_P^T(x) + F_P(x) E_P^T(x) = 0, \quad \forall x \in \mathcal{X} \\ \text{rank} \left[F_S(x) \vdots F_R(x) \vdots F_P(x) \vdots E_S(x) \vdots E_R(x) \vdots E_P(x) \right] = \dim \mathcal{F} \end{array} \tag{28}$$

Remark 3.8 Strictly speaking the flow and effort variables $\dot{x}(t) = -f_S(t)$, respectively $\frac{\partial H}{\partial x}(x(t)) = e_S(t)$, are not living in the constant linear space $\mathcal{F}_S$, respectively $\mathcal{F}_S^*$, but instead in the tangent spaces $T_{x(t)}\mathcal{X}$, respectively co-tangent spaces $T^*_{x(t)}\mathcal{X}$, to the state space manifold $\mathcal{X}$. This is formalized in the definition of a *non-constant Dirac structure on a manifold*; see [3, 5, 4, 21].

By the power-conservation property of a Dirac structure (cf. (10)) it follows directly that any implicit PCHD system satisfies the energy-inequality

$$\begin{aligned} \frac{dH}{dt}(x(t)) &= < \frac{\partial H}{\partial x}(x(t)) | \dot{x}(t) >= \\ &= -e_R^T(t) S e_R(t) + e_P^T(t) f_P(t) \leq e_P^T(t) f_P(t), \end{aligned} \tag{29}$$

showing passivity if $H \geq 0$. The *algebraic constraints* that are present in the implicit system (27) are expressed by the subspace P_1, and the Hamiltonian H. In fact, since the Dirac structure $\mathcal{D}$ is modulated by the x-variables, also the subspace P_1 is modulated by the x-variables, and thus the effort variables e_S, e_R and e_P necessarily satisfy $(e_S, e_R, e_P) \in P_1(x),\ x \in \mathcal{X}$, and thus, because of (21),

$$e_S \in \text{ Im } F_S^T(x), e_R \in \text{ Im } F_R^T(x), e_P \in \text{ Im } F_P^T(x). \tag{30}$$

The second and third inclusions entail the expression of e_R and e_P in terms of the other variables, while the first inclusion determines, since $e_S = \frac{\partial H}{\partial x}(x)$, the following algebraic constraints on the state variables

$$\frac{\partial H}{\partial x}(x) \in \text{ Im } F_S^T(x). \tag{31}$$

The *Casimir functions* $C : \mathcal{X} \to \mathbb{R}$ of the implicit system (27) are determined by the subspace $G_1(x)$. Indeed, necessarily $(f_S,\ f_R,\ f_P) \in G_1(x)$, and thus by (21)

$$f_S \in \text{ Im } E_S^T(x),\ f_R \in \text{ Im } E_R^T(x),\ f_P \in \text{ Im } E_P^T(x). \tag{32}$$

Since $f_S = -\dot{x}(t)$, the first inclusion yields the *flow constraints* $\dot{x}(t) \in \text{ Im } E_S^T(x(t)), \quad t \in \mathbb{R}$. Thus $C : \mathcal{X} \to \mathbb{R}$ is a Casimir function if $\frac{dC}{dt}(x(t)) = \frac{\partial^T C}{\partial x}(x(t))\dot{x}(t) = 0$ for all $\dot{x}(t) \in \text{ Im } E_S^T(x(t))$. Hence $C : \mathcal{X} \to \mathbb{R}$ is a Casimir of the implicit PCHD system (26) if it satisfies the set of p.d.e.'s

$$\frac{\partial C}{\partial x}(x) \in \text{ Ker } E_S(x) \tag{33}$$

Remark 3.9 Note that $C : \mathcal{X} \to \mathbb{R}$ satisfying (33) is a Casimir function of (26) in a *strong sense*: it is a dynamical invariant $\left(\frac{dC}{dt}(x(t)) = 0\right)$ for *every* port behavior and every resistive relation (25).

Example 3.10 [4, 23, 25] Consider a mechanical system with k degrees of freedom, locally described by k configuration variables $q = (q_1, \ldots, q_k)$. Suppose that there are constraints on the generalized velocities $\dot{q}$, described as $A^T(q)\dot{q} = 0$, with $A(q)$ a $r \times k$ matrix of rank r everywhere (that is, there are r independent kinematic constraints). This leads to the following *constrained Hamiltonian equations*

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p}(q, p) \\ \dot{p} &= -\frac{\partial H}{\partial q}(q, p) + A(q)\lambda + B(q)u \\ y &= B^T(q)\frac{\partial H}{\partial p}(q, p) \\ 0 &= A^T(q)\frac{\partial H}{\partial p}(q, p) \end{aligned} \tag{34}$$

where $B(q)u$ are the external forces (controls) applied to the system, for some $k \times m$ matrix $B(q)$, while $A(q)\lambda$ are the *constraint forces*. The Lagrange multipliers $\lambda(t)$ are uniquely determined by the requirement that the constraints $A^T(q(t))\dot{q}(t) = 0$ have to be satisfied for all t. One way of proceeding with these equations is to *eliminate* the constraint forces, and to *reduce* the equations of motion to the constrained state space $\mathcal{X}_c = \{(q, p) \mid A^T(q)\frac{\partial H}{\partial p}(q, p) = 0\}$. In [24] it has been shown that this leads to an explicit port-controlled Hamiltonian system (1). Alternatively, the constrained Hamiltonian equations (34) can be viewed as an *implicit* port-controlled Hamiltonian system, with respect to the Dirac structure $\mathcal{D}$, given in constrained input-output representation (14) by

$$
\begin{aligned}
&\mathcal{D} = \{(f_S, f_P, e_S, e_P) | 0 = A^T(q)e_S,\ e_P = B^T(q)e_S,\\
&-f_S = \begin{bmatrix} 0 & I_k \\ -I_k & 0 \end{bmatrix} e_S + \begin{bmatrix} 0 \\ A(q) \end{bmatrix} \lambda + \begin{bmatrix} 0 \\ B(q) \end{bmatrix} f_P,\ \lambda \in \mathbb{R}^r\}
\end{aligned}
\tag{35}
$$

In this case, the algebraic constraints on the state variables (q, p) are given as $A^T(q)\frac{\partial H}{\partial p}(q, p) = 0$, while the Casimir functions C are determined by the equations

$$
\frac{\partial^T C}{\partial q}(q)\dot{q} = 0, \qquad \text{for all } \dot{q} \text{ satisfying } A^T(q)\dot{q} = 0. \tag{36}
$$

Hence, finding Casimir functions amounts to *integrating* the kinematic constraints $A^T(q)\dot{q} = 0$.

Remark 3.11 For a proper notion of *integrability* of non-constant Dirac structures, generalizing the Jacobi identity for the structure matrix $J(x)$, we refer e.g. to [4]. For example, the Dirac structure (35) is integrable if and only if the kinematic constraints are *holonomic.*

In principle, the theory presented e.g. in the companion paper [22] for stabilization of *explicit* port-controlled Hamiltonian systems can be directly extended, mutatis mutandis, to *implicit* port-controlled Hamiltonian system. In particular, the standard feedback interconnection of an implicit port-controlled Hamiltonian system P with port variables f_P, e_P (the "plant") with another implicit port-controlled Hamiltonian system with port variables f_P^C, e_P^C (the "controller"), via the interconnection relations

$$
\begin{aligned}
f_P &= -e_P^C + f^{\text{ext}}\\
f_P^C &= e_P + e^{\text{ext}}
\end{aligned}
\tag{37}
$$

is readily seen to result in a closed-loop implicit port-controlled Hamiltonian system with port variables $f^{\text{ext}}, e^{\text{ext}}$. Furthermore, as in the explicit case, the Hamiltonian of this closed-loop system is just the *sum* of the Hamiltonian of the plant PCHD system and the Hamiltonian of the controller PCHD system. Finally, the Casimir analysis for the closed-loop system can be performed along the same lines as before.

4 Scattering representation and relation with L_2-gain

In this section we indicate the close connection of the theory of port-controlled Hamiltonian systems with the L_2-gain (or nonlinear H_∞ norm) of nonlinear systems via *scattering*, see [21, 15]. First we deal with the scattering representation of power-conserving interconnections. Let us consider as before an ℓ-dimensional linear space $\mathcal{F}$, with the canonically defined symmetric bilinear form $<,>_{\mathcal{F}\times\mathcal{F}^*}$ on $\mathcal{F} \times \mathcal{F}^*$ given in (8). It follows immediately that $<,>_{\mathcal{F}\times\mathcal{F}^*}$ has singular values $+1$ (multiplicity ℓ) and -1 (multiplicity ℓ). Thus, we can intrinsically define the ℓ-dimensional subspace $V \subset \mathcal{F} \times \mathcal{F}^*$

as a (generally not uniquely defined) *positive space* of $<,>_{\mathcal{F}\times\mathcal{F}^*}$, and the ℓ-dimensional subspace $Z \subset \mathcal{F}\times\mathcal{F}^*$ as a *negative space* of $<,>_{\mathcal{F}\times\mathcal{F}^*}$, that is

$$\begin{aligned} V &= \{(f,e)\in\mathcal{F}\times\mathcal{F}^* \mid <(f,e),(f,e)>_{\mathcal{F}\times\mathcal{F}^*}\geq 0\} \\ Z &= \{(f,e)\in\mathcal{F}\times\mathcal{F}^* \mid <(f,e),(f,e)>_{\mathcal{F}\times\mathcal{F}^*}\leq 0\} \end{aligned} \tag{38}$$

Then the subspaces V and Z have the complementarity property

$$\mathcal{F}\times\mathcal{F}^* = V \oplus Z \tag{39}$$

Furthermore, by restricting $<,>_{\mathcal{F}\times\mathcal{F}^*}$ to V we obtain an inner product $<,>_V$ on V, and by restricting $-<,>_{\mathcal{F}\times\mathcal{F}^*}$ to Z we obtain an inner product $<,>_Z$ on Z.
Any element $(f,e)\in\mathcal{F}\times\mathcal{F}^*$ can thus also be represented as a pair $(v,z)\in V\oplus Z$. This is called a *scattering* representation of (f,e). Let $(f_1,e_1)=(v_1,z_1)$ and $(f_2,e_2)=(v_2,z_2)$, then by orthogonality of V and Z with respect to $<,>_{\mathcal{F}\times\mathcal{F}^*}$ we obtain from (8) the fundamental relation

$$<e_1|f_2> + <e_2|f_1> = <v_1,v_2>_V - <z_1,z_2>_Z \tag{40}$$

In particular, taking $(f_1,e_1)=(f_2,e_2)=(f,e)$, we obtain for $(f,e)=(v,z)\in V\oplus Z$

$$2<e|f> = ||v||_V^2 - ||z||_Z^2 \tag{41}$$

where $||\ ||_V,\ ||\ ||_Z$ are the norms on $V, Z,$ defined by $<,>_V$, respectively $<,>_Z$.
A coordinate expression for the scattering representation is obtained as follows. Take any basis $b_1,\cdots,b_\ell,b_1^*,\cdots,b_\ell^*$ for $\mathcal{F}\times\mathcal{F}^*$. A corresponding basis for V and Z is given as

$$\begin{aligned} V &= \operatorname{span}\left\{\left(\frac{b_i}{\sqrt{2}},\frac{b_i^*}{\sqrt{2}}\right), i=1,\cdots,\ell\right\} \\ Z &= \operatorname{span}\left\{\left(\frac{-b_i}{\sqrt{2}},\frac{b_i^*}{\sqrt{2}}\right), i=1,\cdots,\ell\right\} \end{aligned} \tag{42}$$

where the factors $\frac{1}{\sqrt{2}}$ have been inserted in order that these bases are *orthonormal* with respect to the intrinsically defined inner products $<,>_V$ and $<,>_Z$. With respect to these bases the relation between the representations $(f,e)\in\mathcal{F}\times\mathcal{F}^*$ and $(v,z)\in V\oplus Z$ is expressed in coordinates as

$$\begin{aligned} v &= \frac{1}{\sqrt{2}}(f+e) \\ z &= \frac{1}{\sqrt{2}}(-f+e) \end{aligned} \tag{43}$$

Now, consider a (constant) Dirac structure $\mathcal{D}$ on $\mathcal{F}$, that is, a linear subspace $\mathcal{D}\subset\mathcal{F}\times\mathcal{F}^*$ with the property $\mathcal{D}=\mathcal{D}^\perp$, with $\perp$ denoting orthogonal complement with respect to $<,>_{\mathcal{F}\times\mathcal{F}^*}$. It follows that $<,>_{\mathcal{F}\times\mathcal{F}^*}$ is *zero* when restricted to $\mathcal{D}$, and thus $\mathcal{D}\cap V=0$ and $\mathcal{D}\cap Z=0$. This implies that the Dirac structure $\mathcal{D}$ can be represented as the *graph* of an *invertible* linear map $O:V\to Z$, that is,

$$\mathcal{D} = \{(f,e)=(v,z)|z=Ov, v\in V\} \tag{44}$$

where $(v,z)\in V\oplus Z$ is the scattering representation of $(f,e)\in\mathcal{F}\times\mathcal{F}^*$. Furthermore, for any $(f_1,e_1),(f_2,e_2)\in\mathcal{D}$, with scattering representation (v_1,z_1), respectively (v_2,z_2), we obtain by (40) and $\mathcal{D}=\mathcal{D}^\perp$

$$0 = <e_1|f_2> + <e_2|f_1> = <v_1,v_2>_V - <z_1,z_2>_Z, \tag{45}$$

implying that

$$< z_1, z_2 >_Z = < Ov_1, Ov_2 >_Z = < v_1, v_2 >_V \tag{46}$$

for all $v_1, v_2 \in V$. Hence, the linear map $O : V \to Z$ is an *inner-product preserving* map from V with inner product $<, >_V$ to Z with inner product $<, >_Z$. Conversely, let $O : V \to Z$ be an inner-product preserving map. If we now *define* $\mathcal{D}$ by (44), then by (45) and (46)

$$0 = < v_1, v_2 >_V - < z_1, z_2 >_Z = < e_1 | f_2 > + < e_2 | f_1 > ,$$

and thus $\mathcal{D} \subset \mathcal{D}^\perp$. Furthermore, because dim $\mathcal{D} = \ell$, we conclude $\mathcal{D} = \mathcal{D}^\perp$, implying that $\mathcal{D}$ is a Dirac structure. *Hence constant Dirac structures $\mathcal{D}$ on $\mathcal{F}$ are in one-to-one correspondence with inner-product preserving linear maps $O : V \to Z$.* We call O the *scattering representation* of $\mathcal{D}$.
A matrix representation of O is obtained as follows. Let as before $b_1, \cdots, b_\ell, b_1^*, \cdots, b_\ell^*$ be a basis for $\mathcal{F} \times \mathcal{F}^*$, and corresponding to this basis let $\mathcal{D}$ be given in kernel representation as

$$\mathcal{D} = \{(f, e) | Ff + Ee = 0\} \tag{47}$$

with F, E square $\ell \times \ell$ matrices satisfying (14).

Proposition 4.1 *([21]) The matrix representation of $O : V \to Z$ is the orthonormal matrix*

$$O = (F - E)^{-1}(F + E) \tag{48}$$

Example 4.2 Let the constant Dirac structure $\mathcal{D}$ be given by a skew-symmetric matrix J, that is, $\mathcal{D} = \{(f, e) | f = Je, \ J = -J^T\}$. Then the scattering representation of $\mathcal{D}$ is the orthonormal matrix

$$O = (I + J)^{-1}(I - J)$$

which is known as the *Cayley transform* of J. The same result holds for *non-constant* Dirac structures, modulated by the state variables x. In this case, the Dirac structure is represented by an orthonormal matrix $O(x)$ depending smoothly on x. For example, the scattering representation of a Dirac structure defined by $J(x) = -J^T(x)$ as above is $O(x) = (I + J(x))^{-1}(I - J(x))$.

Now, let us consider an (explicit) port-controlled Hamiltonian system

$$\begin{aligned} \dot{x} &= J(x)\frac{\partial H}{\partial x}(x) + g(x)u \\ y &= g^T(x)\frac{\partial H}{\partial x}(x) \end{aligned} \tag{49}$$

with an underlying Dirac structure defined by $J(x)$ and $g(x)$ (cf. Example 3.6), with respect to the power-variables $(f_S, e_S, \ f_P, e_P) = (-\dot{x}, \frac{\partial H}{\partial x}(x), u, y)$. (Similar considerations hold for port-controlled Hamiltonian systems with *dissipation* and *implicit* port-controlled Hamiltonian systems.) Consider the scattering representation of $(f_P, e_P) = (u, y)$ given by (43). The inverse of this transformation is $u = \frac{1}{\sqrt{2}}(v - z)$, $y = \frac{1}{\sqrt{2}}(v + z)$, which by substitution in (49) yields

$$\begin{aligned} \dot{x} &= \left[J(x) - g(x)g^T(x)\right]\frac{\partial H}{\partial x}(x) + \sqrt{2}g(x)v \\ z &= \sqrt{2}g^T(x)\frac{\partial H}{\partial x} - v \end{aligned} \tag{50}$$

Note that, compared with (49), "dissipation" has been inserted here in two ways: (i) by a dissipation structure $g(x)g^T(x) \geq 0$ in the dynamical equations, (ii) by a negative unity feedthrough from v to z.

The energy-balancing property (3) for the PCH system (49) then translates, using (41), into

$$\frac{dH}{dt}(x(t)) = u^T(t)y(t) = \frac{1}{2}||v||_V^2 - \frac{1}{2}||z||_Z^2 \tag{51}$$

which expresses the fact that the PCH system in scattering representation (50) has L_2-gain equal to 1. In fact, it shows that the system (50) is *inner* or *all-pass.* This provides a fundamental link between PCH(D) systems with systems that are inner or have L_2-gain equal or less than one.

5 Conclusions

It has been shown how the framework of port-controlled Hamiltonian systems as discussed in the companion paper [22] can be naturally extended to systems with algebraic constraints (DAE's), leading to the notion of implicit port-controlled Hamiltonian systems. The key feature in this generalization is the geometric notion of a Dirac structure, capturing general power-conserving interconnections. It has been demonstrated that the existence of Casimir functions (dynamic invariants) and algebraic constraints can be directly translated into properties of the Dirac structure. In [4, 23, 20] it has been shown that any *composition* of Dirac structures in a power-conserving way again leads to a Dirac structure, thus leading to a completely *modular* notion of port-controlled Hamiltonian systems.
In this paper we have concentrated on *constant* Dirac structures, leaving aside the interesting integrability issues connected to non-constant Dirac structures, see e.g. [4, 23]. Currently, the framework of port-controlled Hamiltonian systems is being extended to *distributed* parameter systems, see [16], where the notion of (an infinite-dimensional) Dirac structure has shown to be indispensable in incorporating energy-flow through the boundary of the spatial domain of the system.
Finally, a geometric definition of scattering has been provided, giving a direct connection of port-controlled Hamiltonian systems with *inner* systems (or, systems having L_2- gain less than one in the case of port-controlled Hamiltonian systems with dissipation). It has been demonstrated how in the scattering representation power-conserving interconnections correspond to *orthonormal* maps. These results provoke new approaches in the design and robust control of (lumped-parameter and distributed-parameter) physical systems, see already [26] for some initial results in this direction.

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