AUTOMATIC CONTROL AND SYSTEM THEORY

ANALYSIS IN THE PHASE PLANE

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Phase plane analysis

Phase plane analysis is one of the most important techniques for studying the behavior of dynamic systems, especially in the nonlinear case, where general methods for computing analytical solution do not exist.

Some comments:

• The response characteristics (relative speed of response) for unforced systems depend on the initial conditions

• Eigenvalue/eigenvector analysis allows us to predict the response characteristics (fast and slow, or stable and unstable) depending on initial conditions

• Another way of obtaining a feel for the effect of initial conditions (and then of the characteristics of the response) is to use a phase-plane plot

• A phase-plane plot for a two-state variable system consists of curves of one state variable versus the other one ($x_1(t)$ vs. $x_2(t)$), where each curve is based on a different initial condition
Phase plane analysis

• Consider a system of linear differential equations \( \mathbf{x}' = A\mathbf{x} \). Its \textit{phase portrait} is a representative set of its solutions, plotted as parametric curves (with \( t \) as the parameter) on the Cartesian plane tracing the path of each particular solution \( (x, y) = (x_1(t), x_2(t)), -\infty < t < \infty \). Similar to a direction field, a phase portrait is a graphical tool to visualize how the solutions of a given system of differential equations would behave in the long run.

• In this context, the Cartesian plane where the phase portrait resides is called the \textit{phase plane}. The parametric curves traced by the solutions are sometimes also called their \textit{trajectories}.

• \textbf{Remark}: It is quite labor-intensive, but it is possible to sketch the phase portrait by hand without first having to solve the system of equations that it represents. Just like a direction field, a phase portrait can be a tool to predict the behaviors of a system’s solutions. To do so, we draw a grid on the phase plane. Then, at each grid point \( \mathbf{x} = (\alpha, \beta) \), we can calculate the solution trajectory’s instantaneous direction of motion at that point by using the given system of equations to compute the tangent / velocity vector, \( \mathbf{x}' \). Namely plug in \( \mathbf{x} = (\alpha, \beta) \) to compute \( \mathbf{x}' = A\mathbf{x} \).
Phase plane analysis – Equilibrium solutions

• An equilibrium solution of the system $x' = Ax$ is a point $(x_1, x_2)$ where $x' = 0$, that is, where $x_1' = x_2' = 0$. An equilibrium solution is a constant solution of the system, and is usually called a critical point.

• For a linear system $x' = Ax$, an equilibrium solution occurs at each solution of the system (of homogeneous algebraic equations) $Ax = 0$. As we have seen, such a system has exactly one solution, located at the origin, if $\det(A) \neq 0$. If $\det(A) = 0$, then there are infinitely many solutions.

• For our purpose, and unless otherwise noted, we will only consider systems of linear differential equations whose coefficient matrix $A$ has nonzero determinant. That is, we will only consider systems where the origin is the only critical point.
Phase plane analysis – Equilibrium solutions

• The critical points of various systems of first order linear differential equations are classified by using their stability. In addition, due to the truly two-dimensional nature of the parametric curves, we will also classify the type of those critical points by the shapes formed by the trajectories about each critical point.

• Comment: The accurate tracing of the parametric curves of the solutions is not an easy task without computers. However, we can obtain very reasonable approximation of a trajectory by using the very same idea behind the direction field, namely the tangent line approximation. At each point \( \mathbf{x} = (x_1, x_2) \) on the plane, the direction of motion of the solution curve that passes through the point is determined by the direction vector (i.e. the tangent vector) \( \mathbf{x}' \), the derivative of the solution vector \( \mathbf{x} \), evaluated at the given point. The tangent vector at each given point can be calculated directly from the given matrix-vector equation \( \mathbf{x}' = A\mathbf{x} \), using the position vector \( \mathbf{x} = (x_1, x_2) \). Like working with a direction field, there is no need to find the solution before performing this approximation.
Phase plane analysis

Example 1: A stable equilibrium point (sink)

\[
\begin{align*}
\dot{x}_1 &= -x_1 \\
\dot{x}_2 &= -5x_2
\end{align*}
\]

\[
\begin{align*}
x_1(t) &= x_1(0) e^{-t} \\
x_2(t) &= x_2(0) e^{-5t}
\end{align*}
\]

Plot of \(x_1\) and \(x_2\) for different initial conditions

The solutions converge to \((0, 0)\) for all initial conditions

The point \((0,0)\) is a stable equilibrium point for the system

\[\text{stable node}\]
Phase plane analysis

Example 2: An unstable equilibrium point

\[
\begin{align*}
\dot{x}_1 &= x_1 \\
\dot{x}_2 &= 5 \ x_2
\end{align*}
\]

\[
\begin{align*}
x_1(t) &= x_1(0) \ e^t \\
x_2(t) &= x_2(0) \ e^{5t}
\end{align*}
\]

Plot of \(x_1\) and \(x_2\) for different initial conditions

The solutions is unstable for any initial condition

\[\text{unstable node}\]
Phase plane analysis

Example 3: An unstable equilibrium point (saddle)

\[
\begin{align*}
\dot{x}_1 &= -x_1 \\
\dot{x}_2 &= 5x_2
\end{align*}
\]

\[
\begin{align*}
x_1(t) &= x_1(0) e^{-t} \\
x_2(t) &= x_2(0) e^{5t}
\end{align*}
\]

Plot of $x_1$ and $x_2$ for different initial conditions

If $x_2(0) = 0$, the solutions converge to $(0, 0)$, otherwise it always $\to \infty$

Axis $x_1$ is a stable subspace, $x_2$ the unstable one

\[\text{saddle point}\]
Phase plane analysis

Example 4: Another saddle point

\[
\begin{align*}
\dot{x}_1 &= 2x_1 + x_2 \\
\dot{x}_2 &= 2x_1 - x_2 \\
\end{align*}
\]

\[
A = \begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix}
\]

\[
\lambda_1 = 2.5616, \quad \lambda_2 = -1.5616
\]

\[
x_{1,2} = \begin{bmatrix} 0.8719 & -0.2703 \\ 0.4896 & 0.9628 \end{bmatrix}
\]

Plot of \(x_1\) and \(x_2\) for different initial conditions

Eigenvector \(x_1\) is an unstable subspace, while \(x_2\) is stable

\[\rightarrow \text{saddle point}\]
Phase plane analysis

Example 5: Another stable point

\[
\begin{align*}
\dot{x}_1 &= -x_1 + x_2 \\
\dot{x}_2 &= -0.5 \ x_2
\end{align*}
\]

\[
A = \begin{bmatrix}
-1 & 1 \\
0 & -0.5
\end{bmatrix}
\]

\[
\lambda_1 = -1, \quad \lambda_2 = -0.5
\]

\[
x_{1,2} = \begin{bmatrix}
1 & 0.8944 \\
0 & 0.4472
\end{bmatrix}
\]

Plot of \(x_1\) and \(x_2\) for different initial conditions

Both eigenspaces \(x_1\) and \(x_2\) are stable
Phase plane analysis

Example 6: A stable point with complex eigenvalues $\rightarrow$ stable spiral

$$A = \begin{bmatrix} 0 & 1 \\ -7.25 & -2 \end{bmatrix}$$

$$\lambda_1 = -1 + 2.5i, \quad \lambda_2 = -1 - 2.5i$$
Phase plane analysis

Example 7: An unstable point with complex eigenvalues $\rightarrow$ unstable spiral

$$A = \begin{bmatrix} 0 & 1 \\ -7.25 & 2 \end{bmatrix} \quad \begin{cases} \lambda_1 = 1 + 2.5i \\ \lambda_2 = 1 - 2.5i \end{cases}$$
Phase plane analysis

Example 8: Stable configurations $\rightarrow$ center (closed trajectories)

$$A = \begin{bmatrix} 0 & 1 \\ -6.25 & 0 \end{bmatrix} \quad \left\{ \begin{array}{l} \lambda_1 = +2.5 \, i, \\ \lambda_2 = -2.5 \, i \end{array} \right.$$
Linear systems

• In conclusion, for linear systems we have only ONE equilibrium point that, depending on the shape of the trajectories is classified as:

  • NODE (or SINK)
    • Stable
    • Unstable

  • SADDLE
    • Stable
    • Unstable

  • SPIRAL
    • Stable
    • Unstable

  • CENTER
Non-linear systems

For nonlinear systems the study of equilibrium points is more complex. As a matter of fact, it is possible that:

1) More **equilibrium points** exist, i.e. more solutions of the equation
   \[ 0 = f(x) \]

2) One (or more) **limit cycle** exists, i.e. closed trajectories in the plane with the property of attracting (or being repulsive) for other trajectories
Non-linear systems

• Nonlinear systems will often have the same general phase-plane behavior as the model linearized about the equilibrium (steady-state) point, when the system is close to that particular equilibrium point.

• Nonlinear systems often have multiple steady-state solutions. Phase-plane analysis of nonlinear systems provides an understanding of which steady-state solution that a particular set of initial conditions will converge to.

• The local behavior (close to one of the steady-state solutions) can be understood from a linear phase-plane analysis of the particular steady-state solution (equilibrium point).
Non-linear systems – Example 1

Consider the system

\[ \begin{align*}
\dot{x}_1 &= x_2(x_1 + 1) \\
\dot{x}_2 &= x_1(x_2 + 3)
\end{align*} \]

There are two equilibrium solutions:

- Equilibrium 1: \( x_{1s} = 0, \quad x_{2s} = 0 \)
- Equilibrium 2: \( x_{1s} = -1, \quad x_{2s} = -3 \)

Linearizing the system we have:

\[ A = \begin{bmatrix} x_{2s} & x_{1s} + 1 \\ x_{2s} + 3 & x_{1s} \end{bmatrix} \]

\[ A = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix} \]

\[ A = \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix} \]

\[ \lambda_1 = -\sqrt{3} \]
\[ \lambda_2 = \sqrt{3} \]

\[ \lambda_1 = -3 \]
\[ \lambda_2 = -1 \]
Non-linear systems – Example 1

• Equilibrium 1: $x = [0, 0]^T$

\[
A = \begin{bmatrix}
0 & 1 \\
3 & 0
\end{bmatrix}
\]

\[
\lambda_1 = -\sqrt{3} \\
\lambda_2 = \sqrt{3}
\]

• The equilibrium point is a saddle

• Eigenvectors:
  
  • $e_1 = [-0.5, \ 0.866]^T$
  
  • $e_2 = [0.5, \ 0.866]^T$
Non-linear systems – Example 1

• Equilibrium 2: \( x = [-1, -3]^T \)

\[
A = \begin{bmatrix}
-3 & 0 \\
0 & -1 \\
\end{bmatrix}
\]

\[
\begin{align*}
\lambda_1 &= -3 \\
\lambda_2 &= -1
\end{align*}
\]

• The equilibrium point is a sink

• Eigenvectors:
  - \( e_1 = [1, 0]^T \)
  - \( e_2 = [0, 1]^T \)
Non-linear systems – Example 1

- Phase-plane plot of the non-linear system
- Two equilibrium points (stable, saddle)
- The linearized model captures the behavior of the system only for areas close to the equilibrium points
Non-linear systems – Example 2

Consider two interacting tanks in series, whose outlet flowrate is a function of the square root of the liquid height $x$ (normally is a linear function), and with an input flow $f$.

**Dynamic model:**

\[
\begin{align*}
\dot{x}_1 &= -\frac{R_1}{A_1} \sqrt{x_1 - x_2} + \frac{1}{A_1} f \\
\dot{x}_2 &= \frac{R_1}{A_2} \sqrt{x_1 - x_2} - \frac{R_2}{A_2} \sqrt{x_2}
\end{align*}
\]

$$A_1 = 5 \text{ m}^2$$
$$A_2 = 10 \text{ m}^2$$
$$R_1 = 2.5 \text{ m}^{2.5} / \text{min}$$
$$R_2 = 5/(6^{1/2}) \text{ m}^{2.5} / \text{min}$$
$$f = 5 \text{ m}^3 / \text{min}$$

*Steady-state: $x_{10} = 10 \text{ m}$, $x_{20} = 6 \text{ m}$*

**Linearized model:**

\[
A = \begin{bmatrix}
-0.1250 & 0.1250 \\
0.0625 & -0.1042
\end{bmatrix}
\]

\[
\lambda_1 = -0.2036, \quad \lambda_2 = -0.0256
\]

\[
x_1 = \begin{bmatrix}
-0.8466 \\
0.5323
\end{bmatrix}, \quad x_2 = \begin{bmatrix}
-0.7827 \\
-0.6224
\end{bmatrix}
\]
Non-linear systems – Example 2

global A1 A2 R1 R2 F A

%%% -- Non linear model
A1 = 5; A2 = 10; R1 = 2.5; R2 = 5/sqrt(6); F = 5;
for h10 = 8:12
    for h20 = 4:8
        x0 = [h10; h20];
        [t,x] = ode45('tank_nl_eqn',[0 200],x0);
        plot(x(:,1),x(:,2),x(1,1),x(1,2),'or'); hold on;
    end
end

%%% -- Linear model
A = [-0.125 0.125;
     0.0625, -0.1042];
for h10 = -2:2
    for h20 = -2:2
        x0 = [h10; h20];
        [t,x] = ode45('tank_eqn',[0 200],x0);
        plot(x(:,1)+10,x(:,2)+6,'r'); hold on;
    end
end

grid
hold off

%%% -- Non linear model
function dx = tank_nl_eqn(t,x);
global A1 A2 R1 R2 F
dx1 = F/A1 - R1/A1*sqrt(x(1)-x(2));
dx2 = R1/A2*sqrt(x(1)-x(2))-R2/A2*sqrt(x(2));
dx = [dx1; dx2];

%%% -- Linear model
function dx = tank_eqn(t,x);
global A
dx = A*x;
Bioreactor: Dynamic model

\[
\begin{align*}
\dot{x}_1 &= (\mu - 0.4)x_1 \quad &x_1 &: \text{biomass concentration} \\
\dot{x}_2 &= (4 - x_2)0.4 - \frac{\mu x_1}{0.4} \quad &x_2 &: \text{substrate concentration} \\
\mu &= \frac{0.53 x_2}{0.12 + x_2} \quad &\text{grow rate}
\end{align*}
\]

Perform a phase-plane analysis