AUTOMATIC CONTROL AND SYSTEM THEORY

STATE FEEDBACK CONTROL

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State Feedback Control

Some comments:

- **The control law**
  - *Open-loop*: the control law is precomputed by considering the initial state, the final state and the model of the system
  - *Closed-loop*: the control law is computed, at each time instant, on the basis of the actual state of the system and the given specifications

- **Feedback control**
  - *Static feedback*: the control action is an algebraic function of the system state
  - *Dynamic feedback*: the control action is computed on the basis of the state of an auxiliary system with proper dynamics

- **Control objective**
  - *Regulation problems*: the control is designed in such a way that the state of the system (perturbed by disturbances or errors) is driven to the origin with a desired velocity
  - *Tracking problems*: the control system is such that the system output follows (or approximates on the basis of some given criteria) a desired trajectory

In the following, the design of *closed loop, static feedback* controllers for *regulation* problems will be addressed
State Feedback Control

- We consider a (continuous- or discrete-time) linear stationary system and we suppose that all the components of the state vector are directly accessible.

\[
\begin{align*}
\dot{x}(t) &= A x(t) + B u(t) \\
y(t) &= C x(t) + D u(t)
\end{align*}
\]

- Let us consider the control law:

\[u(t) = Kx(t) + v(t), \quad K : m \times n\]
State Feedback Control

• By applying this control law we obtain:

\[
\dot{x}(t) = (A + BK) \, x(t) + B \, v(t)
\]

\[
y(t) = (C + DK) \, x(t) + D \, v(t)
\]

• **NOTE:** An alternative control law is given by the feedback of the output \(y(t)\) instead of the state \(x(t)\).

\[
u(t) = K_1 y(t) + v(t)
\]

• For systems with one input and one output only \((m = p = 1)\), \(K_1\) is a scalar quantity, whereas \(K\) (state feedback) is a vector with \(n\) components.

• Therefore, more degrees of freedom are available in the design of the controller in the case of state feedback.

• It follows that better performance can be achieved (the dynamic properties of the system can be significantly changed).
State Feedback Control

Feedback control invariance property

• If $S_k$ is the system obtained from $S$ by means of state feedback control, then the two systems present the same reachability subspace.

$$\forall \; K \in \mathbb{R}^{m \times n} \quad \Rightarrow \quad \mathcal{V}_{S_k}^+ = \mathcal{V}_S^+$$

• The eigenvalues of the unreachable subspace are not affected by the state feedback. It is possible to verify this property by computing the standard reachability form of the system $S$. By considering $K = [K_1 \; K_2]$, the controlled system becomes:

$$A + BK = \begin{bmatrix} A_{11} + B_1 K_1 & A_{12} + B_1 K_2 \\ 0 & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$$

N.B. $K_1 : m \times \rho$, $K_2 : m \times (n - \rho)$

$\rho = \text{Dimension of the reachable subspace}$
Controllable Canonical Form

- We consider the linear stationary system \( S = (A, b, C, d) \) with one input only

\[
\begin{align*}
\dot{x}(t) &= A x(t) + b u(t) \\
y(t) &= C x(t) + d u(t)
\end{align*}
\]

- **Property:** The system \( S \) is reachable iff it is equivalent to the system \( S_c = (A_c, b_c, C_c, d_c) \) in controllable canonical form (or reachable canonical form), that is a system whose matrices \( A_c \) and \( b_c \) have the structure

\[
A_c = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-\alpha_0 & -\alpha_1 & \ldots & -\alpha_{n-1}
\end{bmatrix} \quad \quad B_c = \begin{bmatrix}
0 \\
0 \\
\vdots \\
1
\end{bmatrix}
\]

where \( \alpha_i \) are the **coefficients of the monic characteristic polynomial** of the matrix \( A \)

\[
q_A(\lambda) = \lambda^n + \lambda^{n-1}\alpha_{n-1} + \cdots + \alpha_0
\]
Controllable Canonical Form

• **NOTE:** even if the input directly affects the last components of the state vector $x$ only, the structure of the matrix $A$ is such that through the input it is possible to modify the whole state $x$:

$$A_c = \begin{bmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-\alpha_0 & -\alpha_1 & -\alpha_2 & \ldots & -\alpha_{n-1}
\end{bmatrix}, \quad B_c = \begin{bmatrix}
0 \\
0 \\
\vdots \\
1
\end{bmatrix}$$

$$\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= x_4 \\
\vdots & = \vdots \\
\dot{x}_{n-1} &= x_n \\
\dot{x}_n &= -\alpha_0 x_1 - \alpha_1 x_2 - \ldots - \alpha_{n-1} x_n + u
\end{align*}$$
The transformation matrix $T$ that pose the system in controllable canonical form $x_c = T x$ is defined as:

$$T = P_1 P_{1c}^{-1}$$

where:

- $P_1$ is the reachability matrix of the system $S$: $P_1 = [b, Ab, A^2b, \ldots, A^{n-1}b]$.
- $P_{1c}$ is the reachability matrix of the system $S_C$: $P_{1c} = [b_C, A_C b_C, A_C^2 b_C, \ldots, A_C^{n-1} b_C]$.
- $(P_{1c})^{-1}$ is structured as:

$$
(P_{1c})^{-1} =
\begin{bmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{n-1} & 1 \\
\alpha_2 & \alpha_3 & \cdots & \cdots & 1 & 0 \\
\alpha_3 & \cdots & \cdots & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\alpha_{n-1} & 1 & \cdots & \cdots & 0 & 0 \\
1 & 0 & \cdots & \cdots & 0 & 0 \\
\end{bmatrix}
$$

$\alpha_i$: coefficients of the monic characteristic polynomial of the matrix $A$

$$q_A(\lambda) = \lambda^n + \lambda^{n-1} \alpha_{n-1} + \cdots + \alpha_0$$
Controllable Canonical Form

\[ A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \]

\[ P_1 = [B, AB, A^2B] = \begin{bmatrix} 0 & 2 & 10 \\ 1 & -1 & 3 \\ 1 & 7 & 29 \end{bmatrix} \]

\[ \text{det}(\lambda I - A) = \lambda^3 - 5\lambda^2 - \lambda + 7 \]

\[ P_{1c}^{-1} = \begin{bmatrix} \alpha_1 & \alpha_2 & 1 \\ \alpha_2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -5 & 1 \\ -5 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \]

\[ T = P_1(P_{1c}^{-1}) = \begin{bmatrix} 0 & 2 & 0 \\ 7 & -6 & 1 \\ -7 & 2 & 1 \end{bmatrix} \]

\[ T^{-1} = \begin{bmatrix} 0.2857 & 0.0714 & -0.0714 \\ 0.5 & 0 & 0 \\ 0 & 0.5 & 0.5 \end{bmatrix} \]

\[ A_c = T^{-1}AT = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -7 & 1 & 5 \end{bmatrix} \]

\[ B_c = T^{-1}B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \]

In controllable canonical form:
Coefficients of the characteristic polynomial
Pole Placement

- **Property:** Let us consider the linear stationary system $S = (A, b, C, d)$ with one input only and fully reachable. For any $n$-order monic polynomial $p(\lambda)$ a matrix $k^T$ of dimension $1 \times n$ exists such that the characteristic polynomial of the state matrix $A + b \, k^T$ of the feedback system coincides with $p(\lambda)$.

- **Proof:** $\alpha_i, i = 1, \ldots, n$ are the coefficients of the characteristic polynomial $q_A(\lambda)$ of the matrix $A$

  $$q_A(\lambda) = \lambda^n + \alpha_{n-1} \lambda^{n-1} + \cdots + \alpha_0$$

  and $d_i, i = 1, \ldots, n$ are the coefficients of the desired polynomial $p(\lambda)$

  $$p(\lambda) = \lambda^n + d_{n-1} \lambda^{n-1} + \cdots + d_0$$

  The couple $(A, b)$ is controllable, therefore a transformation matrix that pose the system in controllable canonical form exists

  $$x_c = T^{-1}x, \quad A_c = T^{-1}AT, \quad b_c = T^{-1}b$$
Pole Placement

- By adopting the control law:
  
  \[ u(t) = k_c^T x_c(t) + v(t) \quad k_c^T = [k_0, k_1, \ldots, k_{n-1}] \]

  we obtain:
  
  \[ \dot{x}_c(t) = (A_c + b_c k_c^T) x_c(t) + b_c v(t) \]

  with:

  \[
  A_c + b_c k_c^T = \begin{bmatrix}
  0 & 1 & 0 & \ldots & 0 \\
  0 & 0 & 1 & \ldots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  k_0 - \alpha_0 & k_1 - \alpha_1 & k_{n-1} - \alpha_{n-1}
  \end{bmatrix}
  \]

  whose characteristic polynomial is:

  \[ q(\lambda) = \lambda^n + (\alpha_{n-1} - k_{n-1})\lambda^{n-1} + \cdots + (\alpha_0 - k_0) \]

  the desired result is obtained by posing: \[ k_i = \alpha_i - d_i \]

  **NOTE:** \[ k^T = k_c^T T^{-1} = k_c^T (P_1 P_{1c}^{-1})^{-1} \]
Pole Placement

• Example: computing the static state feedback matrix $u(t) = k^T x(t)$ such that the eigenvalues of the closed-loop system are -1, -2, -2.

$$\dot{x}(t) = Ax(t) + b u(t) \quad A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad P_1 = \begin{bmatrix} b & Ab & A^2b \end{bmatrix} = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

The system is fully reachable since $\text{rank}(P_1) = 3$

The characteristic polynomial of the matrix $A$ and the desired one are respectively

$$p_A(\lambda) = \text{det}(\lambda I - A) = \text{det}\left( \begin{bmatrix} \lambda - 1 & -2 & 0 \\ 0 & \lambda & -1 \\ 0 & -1 & \lambda \end{bmatrix} \right) = \lambda^3 - \lambda^2 - \lambda + 1$$

$$\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = -1$$

$$p(\lambda) = (\lambda + 2)^2(\lambda + 1) = \lambda^3 + 5\lambda^2 + 8\lambda + 4$$

It follows

$$k_c^T = \begin{bmatrix} \alpha_0 - d_0, & \alpha_1 - d_1, & \alpha_2 - d_2 \end{bmatrix}$$

$$= \begin{bmatrix} -3, & -9, & -6 \end{bmatrix}$$
Pole Placement

- The vector $k^T$ is obtained from
  \[ k^T = k_c^T T^{-1} = k_c^T (P_1 P_{1c}^{-1})^{-1} \]

\[
P_1 = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad P_{1c}^{-1} = \begin{bmatrix} -1 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}
\]

\[
k^T = [-3, \ -9, \ -6] \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -1 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}
\]

\[
= [-3, \ -9, \ -6] \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}^{-1}
\]

\[
= [-3, \ -9, \ -6] \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \frac{1}{2}
\]

\[
= [-9, \ -6, \ 3]
\]

From the charact. Pol. of $A$

\[
p_A(\lambda) = \lambda^3 - \lambda^2 - \lambda + 1
\]

\[
P_{1c}^{-1} = \begin{bmatrix} \alpha_1 & \alpha_2 & 1 \\ \alpha_2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}
\]
Pole Placement

- Verification:

\[
A' = (A + bk^T) = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} -9 \\ -6 \\ 3 \end{bmatrix} \\
= \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} -9 & -6 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -9 & -6 & 3 \end{bmatrix} \\
= \begin{bmatrix} -8 & -4 & 3 \\ 0 & 0 & 1 \\ -9 & -5 & 3 \end{bmatrix}
\]

\[p_{A'}(\lambda) = \lambda^3 + 5\lambda^2 + 8\lambda + 4\]

\[
A_c = T^{-1}AT = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}
\]

\[
A'_c = T^{-1}A'T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -8 & -5 \end{bmatrix}
\]
Pole Placement

\[
A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}
\]

\[
A' = \begin{bmatrix} -8 & -4 & 3 \\ 0 & 0 & 1 \\ -9 & -5 & 3 \end{bmatrix}
\]

Unstable state behavior

Stable state behavior
Observable Canonical Form

**Property:** the system $S = (A, c)$ with one output only is fully observable iff it is algebraically equivalent to the system $S_o = (A_o, c_o)$ in observable canonical form, that is a system whose matrices $A_o$ and $c_o$ are structured as:

$$A_o = \begin{bmatrix} 0 & 0 & \ldots & 0 & -\alpha_0 \\ 1 & 0 & \ldots & 0 & -\alpha_1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & -\alpha_{n-1} \end{bmatrix}$$

$$c_o = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

where $\alpha_i$ are the coefficients of the monic characteristic polynomial of the matrix $A$

$$q_A(\lambda) = \lambda^n + \lambda^{n-1}\alpha_{n-1} + \cdots + \alpha_0$$

**NOTE:** The observable canonical form is obtained from the controllable canonical form of the dual system
The observability matrices $P_2$ and $P_{2o}$ of the original system and of the system in observable canonical form are related by

$$P_{2o} = P_2 \, T$$

where $T$ is the transformation matrix $x = T \, x_o$ that pose the original system in observer canonical form.

$$P_2 \, T = \begin{bmatrix} c \\ cA \\ cA^2 \\ \vdots \\ cA^{n-1} \end{bmatrix} \quad T = \begin{bmatrix} cT \\ cT \, (T^{-1} \, AT) \\ cT \, (T^{-1} \, A^2T) \\ \vdots \\ cT \, (T^{-1} \, A^{n-1}T) \end{bmatrix} = \begin{bmatrix} c_o \\ c_o \, A_o \\ c_o \, A_o^2 \\ \vdots \\ c_o \, A_o^{n-1} \end{bmatrix} = P_{2o}$$

$\longrightarrow \quad T = P_2^{-1} \, P_{2o} \quad \longrightarrow \quad T^{-1} = P_{2o}^{-1} \, P_2$

**NOTE:**

$$A' = T^{-1} \, AT \quad B' = T^{-1} \, B$$

$$C' = CT \quad D' = D$$
If the system state is not known, or not directly accessible for the measurement, the design of a “system” that allows to reconstruct the evolution of the state $x(t)$ on the basis of the system initial conditions, inputs (and outputs) is needed.

This “system” is named state estimator (or observer).

Different estimators:
- **Open-loop**: the estimation is based on the knowledge of the initial cond. and the input
- **Closed-loop**: the estimation is based also on the knowledge of the output
- **Reduced-order**: that gives a non-redundant estimation of the state $x(t)$
The open-loop estimator is a copy of the original system:

\[
\begin{align*}
\dot{x}(t) &= A \hat{x}(t) + B u(t) \\
\hat{x}(t) &= x(t) - \hat{x}(t)
\end{align*}
\]

If the estimation error is defined as \( e(t) = x(t) - \hat{x}(t) \),

\[
\dot{e}(t) = \dot{x}(t) - \dot{\hat{x}}(t) = Ax(t) + Bu(t) - A\hat{x}(t) - Bu(t) = Ae(t)
\]

The error dynamics depends on the eigenvalues of \( A \):

- If the system is unstable, the estimation diverges
- The convergence velocity cannot be changed
Open Loop State Estimator

Observer: a “copy” of the system

\[ A = \begin{bmatrix} -0.5 & 0 \\ -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \]

\[ x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad x_o(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

The initial state is unknown.
Open Loop State Estimator

\[ A = \begin{bmatrix} -0.5 & 0 \\ -1 & 0.02 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \]

\[ x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad x_o(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]
To solve the two drawbacks previously highlighted, it is possible to adopt a closed-loop estimator, that is named *identity estimator* (its state asymptotically converges to the actual system state)

\[ \dot{x}(t) = (A + LC)x(t) + Bu(t) - Ly(t) \]

where \( A, B, C \) are (copies of) the matrices of the “observed” system and \( L \) is a suitably designed matrix.

In this case, the state estimation error dynamics is:

\[
\dot{e}(t) = \dot{x}(t) - \dot{x}(t) = Ax(t) + Bu(t) - (A + LC)x(t) + LCx - Bu(t)
\]
\[
= (A + LC)e(t)
\]
Closed Loop State Estimator

\[
\dot{e}(t) = \hat{x}(t) - \hat{x}(t) = Ax(t) + Bu(t) - (A + LC)\hat{x}(t) + LCx - Bu(t) = (A + LC)e(t)
\]

- The state estimation error dynamics is defined by the eigenvalues of the matrix \((A + LC)\).

- Thanks to the properties of dual systems, the characteristic polynomial of the matrix \((A + LC)\) can be arbitrarily assigned iff the couple \((A^T, C^T)\) is fully reachable, that is if the couple \((A, C)\) is fully observable.

- To compute the matrix \(L\), the couple \((A^T, C^T)\) and the techniques for the eigenvalues allocation in the case of fully reachable systems are considered, obtaining in this way the matrix \(L^T\) (design by duality with the state feedback problem).

- If the couple \((A, C)\) is not fully observable, the state estimation error dynamics characterized by the matrix \((A + LC)\) presents some fixed modes that cannot be changed. In this case, the synthesis of an asymptotic state estimator is possible only if the unobservable part of the system is asymptotically stable.

\[
L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 \\ 0 \end{bmatrix} \quad \Rightarrow \quad (A + LC) = \begin{bmatrix} A_{11} + L_1C_1 & 0 \\ A_{21} + L_2C_1 & A_{22} \end{bmatrix}
\]
Closed Loop State Estimator

- In the design of the state estimator, given an \( n \)-order system \( S \), a new \( n \)-order dynamic system \( S_s \) is defined (the estimator), obtaining therefore an “overall” \( 2n \)-order system.

- In general, part of the state vector can be directly computed analyzing the output \( y = C x \), if the matrix \( C \) is known.

- It is then possible to define a state estimator that reconstructs only the components of the state vector \( x \) that cannot be directly computed from the output: this state estimator is called reduced order estimator (or observer).
Closed Loop State Estimator

\[ A = \begin{bmatrix} -0.5 & 0 \\ -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \]

\[ x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad x_o(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad L = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \]
Closed Loop State Estimator

\[ A = \begin{bmatrix} -0.5 & 0 \\ -1 & 0.02 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \]

\[
x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad x_o(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad L = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}
\]
Regulator Design

- We define as **regulator** the system composed by the union of the **state estimator** and of the **static state feedback** $K$

![Diagram of regulator design](image)
Regulator Design

- The equations of the overall system are:

\[
\begin{align*}
\dot{x}(t) &= A \, x(t) + B \, u(t) \\
y(t) &= C \, x(t) \\
u(t) &= v(t) + K\hat{x}(t) \\
\dot{\hat{x}}(t) &= A \, \hat{x}(t) + B \, u(t) + L(C\hat{x}(t) - y(t))
\end{align*}
\]

that, by some simple calculation, can be rewritten as:

\[
\begin{align*}
\dot{x}(t) &= A \, x(t) + B \, K \, \hat{x}(t) + B \, v(t) \\
y(t) &= C \, x(t) \\
\dot{\hat{x}}(t) &= (A + L \, C + B \, K) \, \hat{x}(t) - L \, Cx(t) + B \, v(t)
\end{align*}
\]
• These equations can be grouped in matrix form in a $2n$-order system, defined as:

$$
\begin{bmatrix}
\dot{x}(t) \\
\hat{x}(t)
\end{bmatrix} =
\begin{bmatrix}
A & B K \\
-L C & (A + L C + B K)
\end{bmatrix}
\begin{bmatrix}
x(t) \\
\hat{x}(t)
\end{bmatrix} +
\begin{bmatrix}
\hat{B} \\
B
\end{bmatrix} v(t)
$$

$$
y(t) =
\begin{bmatrix}
\tilde{C} \\
C & 0
\end{bmatrix}
\begin{bmatrix}
x(t) \\
\hat{x}(t)
\end{bmatrix}
$$

$$
\dot{x}(t) = \tilde{A} \hat{x}(t) + \tilde{B} v(t)
$$

$$
y(t) = \tilde{C} \hat{x}(t)
$$

• With the aim of highlighting some properties of this system, we apply a state transformation defined as:

$$
x' = T \tilde{x}, \quad T = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix}, \quad T^{-1} = T
$$
Regulator Design

- It is important to note that the introduced transformation

\[ x' = T\bar{x}, \quad T = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix}, \quad \Rightarrow \quad x' = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} = \begin{bmatrix} x \\ x - \hat{x} \end{bmatrix} \]

generates a state vector \( x' \) that presents the state of the original system as first component, and the difference between \( x \) and its estimation (state estimation error) as second component.

- For what is related to the system matrices, we obtain:

\[
A' = \begin{bmatrix} (A + BK) & -BK \\ 0 & (A + LC) \end{bmatrix}, \quad B' = \begin{bmatrix} B \\ 0 \end{bmatrix}
\]

\[
C' = \begin{bmatrix} C \\ 0 \end{bmatrix}
\]
Regulator Design

• It is important to note that the presence of the state estimator does not modify the input-output relation of the system: the transfer function (matrix) of the overall system does not change if the estimated state is considered instead of the actual state $x$.

• From

$$A' = \begin{bmatrix} (A + BK) & -BK \\ 0 & (A + LC') \end{bmatrix}, \quad B' = \begin{bmatrix} B \\ 0 \end{bmatrix}$$

$$C' = \begin{bmatrix} C \\ 0 \end{bmatrix}$$

it follows

$$H(s) = \bar{C}(sI - \bar{A})^{-1}\bar{B} = C'(sI - A')^{-1}B'$$

$$= \begin{bmatrix} C \\ 0 \end{bmatrix} \begin{bmatrix} [sI - (A + BK)]^{-1} & \ast \ast \\ 0 & [sI - (A + LC')]^{-1} \end{bmatrix} \begin{bmatrix} B \\ 0 \end{bmatrix}$$

$$= C [sI - (A + BK)]^{-1} B$$

It does not depend on the estimator!
Separation Property

- In the design of the state feedback, only the eigenvalues of the reachable part of the system can be modified by the relation \((A + BK)\).

- In the same way, it is possible to verify that, if the system state is not directly accessible, a state estimator can be designed in which the dynamics of the observable part can be arbitrarily assigned by suitably selecting the eigenvalues of \((A+LC)\).

- We have also verified that, in case a state estimator is necessary for the regulator implementation, the introduction (and the dynamics) of the state estimator does not affect the design of the state feedback (and vice versa).
Separation Property

• The *separation property* holds. The design of:
  • The state feedback, that is the allocation of the eigenvalues of \((A+BK)\)
  • The state estimator, that is the allocation of the eigenvalues of \((A+LC)\)
can be carried out independently.

• The following property holds:

\[
det(sI - A'') = det\left(\begin{bmatrix}
(sI - A - B K) & -B K \\
0 & (sI - A - L C)
\end{bmatrix}\right)
\]

\[
= det(sI - A - B K) \cdot det(sI - A - L C)
\]

Therefore the eigenvalues of the controller and of the estimator can be independently assigned.
Regulator Design

- **Example:** inverted pendulum position control

  State: angle $\theta$ of the pendulum and its velocity
  Output: angle $\theta$

  \[
  \begin{align*}
  \dot{x}_1(t) &= x_2(t) = \dot{\theta} \\
  \dot{x}_2(t) &= g \frac{ml}{J + ml^2} \sin x_1 - \frac{ml}{J + ml^2} \frac{1}{M} u \cos x_1
  \end{align*}
  \]

- By linearizing at the equilibrium point: $x_1 = x_2 = 0$ per $u = 0$

\[
\begin{align*}
\dot{x}(t) &= Ax + Bu \\
y(t) &= Cx
\end{align*}
\]

\[
A = \begin{bmatrix} 0 & 1 \\ g & \alpha \end{bmatrix}, \quad 
B = \begin{bmatrix} 0 \\ -\frac{\alpha}{M} \end{bmatrix}, \quad 
C = \begin{bmatrix} 1 & 0 \end{bmatrix}
\]

\[
P_1 = \begin{bmatrix} 0 & -\frac{\alpha}{M} \\ -\frac{\alpha}{M} & 0 \end{bmatrix}, \quad 
P_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

\[
\text{rank}\{P_1\} = 2, \quad \text{rank}\{P_2\} = 2
\]

N.B.: $\lambda_{1,2} = \pm \sqrt{g\alpha}!$

\[
K = \begin{bmatrix} k_1 & k_2 \end{bmatrix} \quad \text{?} \quad L = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \quad \text{?}
\]
Regulator Design

- **Separation property:**
  - **Regulator:** since \((A, B)\) is fully reachable, the eigenvalues of \((A + BK)\) can be arbitrarily assigned

\[
det(sI - A - BK) = s^2 + (\lambda_1 + \lambda_2)s + \lambda_1 \lambda_2 = (s + \lambda_1)(s + \lambda_2)
\]

\[
A + BK = \begin{bmatrix}
0 & 1 \\
g\alpha - \frac{\alpha}{M}k_1 & -\frac{\alpha}{M}k_2
\end{bmatrix}
\]

- **State estimator:** since \((A, C)\) is fully observable, even if also \(x_1\) is directly available from the output, the eigenvalues of \((A + LC)\) can be arbitrarily assigned

\[
det(sI - A - LC) = (s + \lambda)^2
\]

\[
A + LC = \begin{bmatrix}
l_1 & 1 \\
g\alpha l_2 & 0
\end{bmatrix}
\]

\[
s^2 - l_1s - (g\alpha + l_2) = s^2 + 2\lambda s + \lambda^2
\]

\[
L = \begin{bmatrix}
-2\lambda \\
\lambda^2 - g\alpha
\end{bmatrix}
\]
Regulator Design

\[ \alpha = 5, \quad M = 1 \]

\[ \lambda_1 = 1, \quad \lambda_2 = 3 \quad \lambda = 2 \]

\[ K = \begin{bmatrix} 10.41 & 0.8 \end{bmatrix}, \quad L = \begin{bmatrix} -4 \\ -53.05 \end{bmatrix} \]

Different initial conditions for the real system and for the state estimator.
Regulator Design

\[ \lambda_1 = 1, \quad \lambda_2 = 3 \]

\[
K = \begin{bmatrix} 10.41 & 0.8 \end{bmatrix}
\]

\[ \lambda = 2 \]

\[
L = \begin{bmatrix} -4 \\ -53.05 \end{bmatrix}
\]

\[ \lambda_1 = 0.1, \quad \lambda_2 = 0.3 \]

\[
K = \begin{bmatrix} 9.876 & 0.266 \end{bmatrix}
\]

\[ \lambda = 2 \]

\[
L = \begin{bmatrix} -4 \\ -53.05 \end{bmatrix}
\]
Regulator Design

\[ \lambda_1 = 1, \quad \lambda_2 = 3 \]

\[ K = \begin{bmatrix} 10.41 & 0.8 \end{bmatrix} \]

\[ L = \begin{bmatrix} -4 \\ -53.05 \end{bmatrix} \]

\[ \lambda = 1 \]

\[ L = \begin{bmatrix} -2 \\ -50.05 \end{bmatrix} \]