AUTOMATIC CONTROL AND SYSTEM THEORY

STATE FEEDBACK CONTROL

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State Feedback Control

Some comments:

- *The control law*
  - *Open-loop:* the control law is precomputed by considering the initial state, the final state and the model of the system
  - *Closed-loop:* the control law is computed, at each time instant, on the basis of the actual state of the system and the given specifications
- *Feedback control*
  - *Static feedback:* the control action is an algebraic function of the system state
  - *Dynamic feedback*: the control action is computed on the basis of the state of an auxiliary system with proper dynamics
- *Control objective*
  - *Regulation problems:* the control is designed in such a way that the state of the system (perturbed by disturbances or errors) is driven to the origin with a desired velocity
  - *Tracking problems:* the control system is such that the system output follows (or approximates on the basis of some given criteria) a desired trajectory

In the following, the design of *closed loop*, *static feedback* controllers for *regulation* problems will be addressed

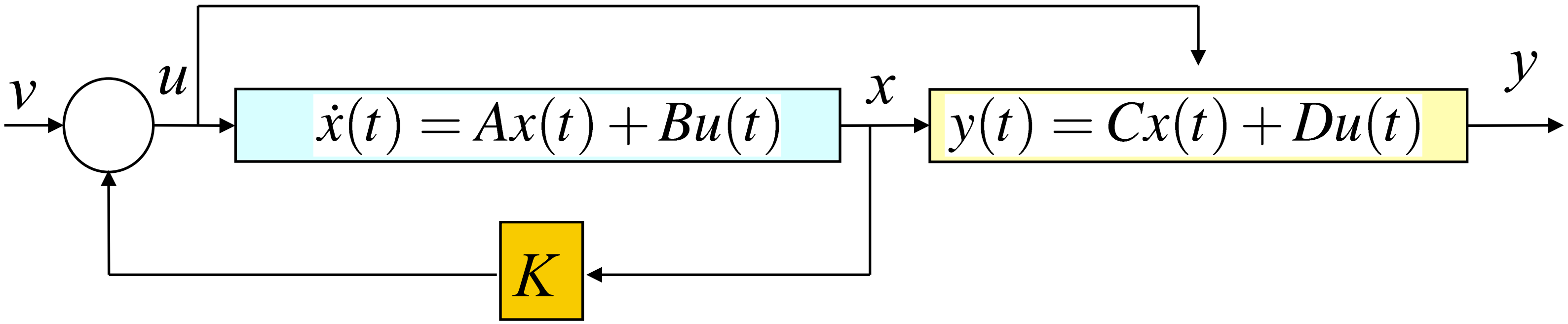
State Feedback Control

- We consider a (continuous- or discrete-time) linear stationary system and we suppose that *all the components of the state vector are directly accessible.*

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t) \\ y(t) &= C\,x(t) + D\,u(t) \end{aligned} \qquad \left(\begin{aligned} x(k+1) &= A\,x(k) + B\,u(k) \\ y(k) &= C\,x(k) + D\,u(k) \end{aligned} \right)$$

- Let us consider the control law:

$$u(t) = Kx(t) + v(t), \qquad K: \ m \times n$$



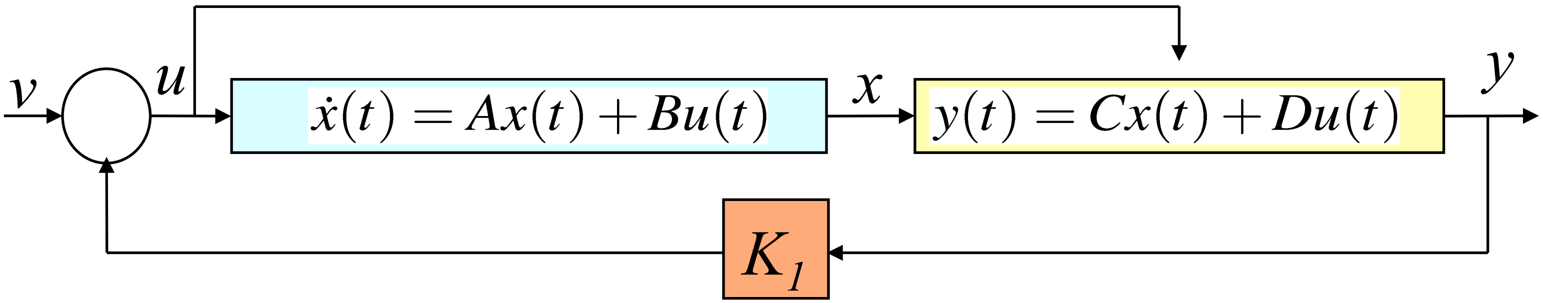
State Feedback Control

- By applying this control law we obtain:

$$\begin{aligned} \dot{x}(t) &= (A+BK)\,x(t)+B\,v(t) \\ y(t) &= (C+DK)\,x(t)+D\,v(t) \end{aligned}$$

- NOTE: An alternative control law is given by the feedback of the output *y(t)* instead of the state *x(t).*

$$u(t) = K_1 y(t) + v(t)$$



- For systems with one input and one output only ($m = p = 1$), K_1 is a scalar quantity, whereas K (state feedback) is a vector with n components.
- Therefore, more degrees of freedom are available in the design of the controller in the case of state feedback.
- It follows that better performance can be achieved (the dynamic properties of the system can be significantly changed).

State Feedback Control

Feedback control invariance property

- If S_k is the system obtained from S by means of state feedback control, then *the two systems present the same reachability subspace.*

$$\forall\, K \in \mathbb{R}^{m\times n} \quad \Longrightarrow \quad \mathcal{V}^+_{S_k} = \mathcal{V}^+_S$$

- *The eigenvalues of the unreachable subspace are not affected by the state feedback*. It is possible to verify this property by computing the standard reachability form of the system S. By considering $K = [K_1 \;\; K_2]$, the controlled system becomes:

$$A + BK = \begin{bmatrix} A_{11} + B_1 K_1 & A_{12} + B_1 K_2 \\ 0 & A_{22} \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$$

N.B. $\quad K_1 : \quad m \times \rho \quad \longrightarrow \quad \rho$ = Dimension of the reachable subspace

$\quad K_2 : \quad m \times (n - \rho)$

Controllable Canonical Form

- We consider the linear stationary system *S = (A, b, C, d)* with one input only

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + b\,u(t) \\ y(t) &= C\,x(t) + d\,u(t) \end{aligned}$$

- ***Property:*** The system *S* is reachable iff it is equivalent to the system $S_c = (A_c, b_c, C_c, d_c)$ in controllable canonical form (or reachable canonical form), that is a system whose matrices A_c and b_c have the structure

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & & & \vdots \\ -\alpha_0 & -\alpha_1 & & & -\alpha_{n-1} \end{bmatrix} \qquad B_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

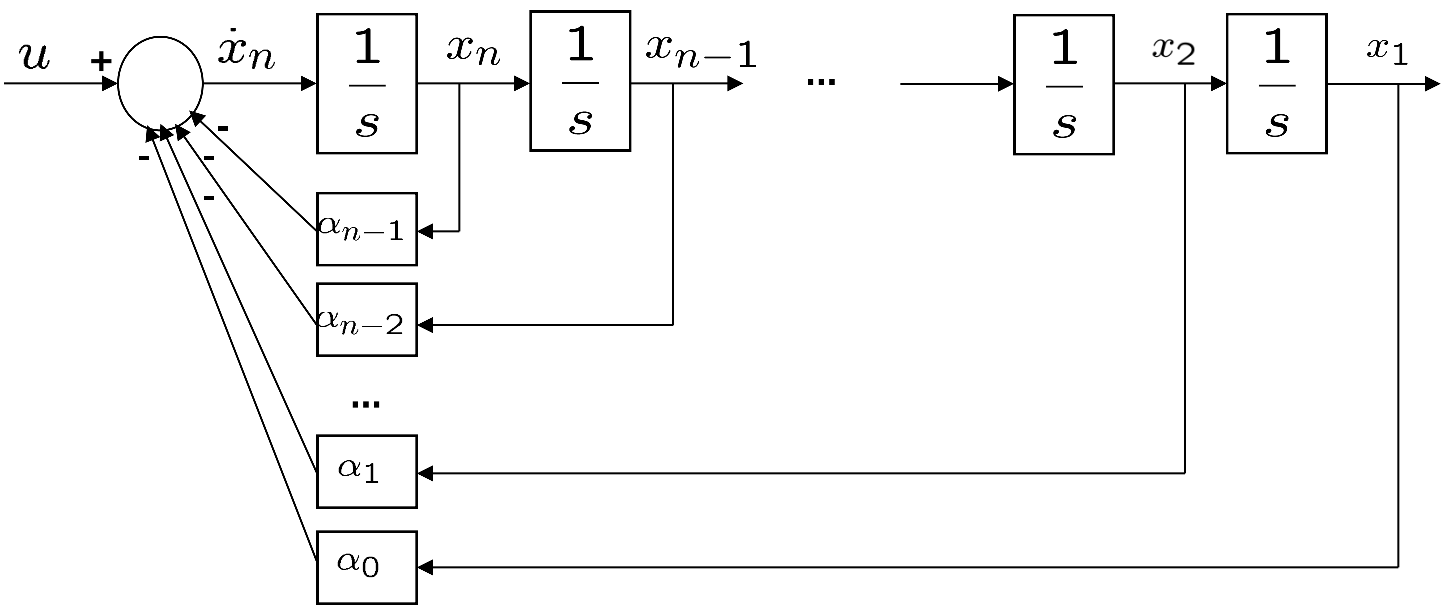
where α_i are the *coefficients of the monic characteristic polynomial* of the matrix *A*

$$q_A(\lambda) = \lambda^n + \lambda^{n-1}\alpha_{n-1} + \dots + \alpha_0$$

Controllable Canonical Form

- **NOTE:** even if the input directly affects the last components of the state vector x only, the structure of the matrix *A* is such that through the input it is possible to modify the whole state *x*:

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & & & \vdots \\ -\alpha_0 & -\alpha_1 & & & -\alpha_{n-1} \end{bmatrix}, \quad B_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \rightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = x_4 \\ \vdots \\ \dot{x}_{n-1} = x_n \\ \dot{x}_n = -\alpha_0 x_1 - \alpha_1 x_2 - \dots - \alpha_{n-1} x_n + u \end{cases}$$



Controllable Canonical Form

- The transformation matrix T that pose the system in controllable canonical form $x_c = T\,x$ is defined as:

$$T = P_1\, P_{1c}^{-1}$$

where:

  - P_1 is the reachability matrix of the system S: $P_1=[b, Ab, A^2b, \ldots, A^{n-1}b]$
  - P_{1C} is the reachability matrix of the system S_C : $P_1=[b_C\,, A_C\, b_C\,, A_C{}^2 b_C\,, \ldots, A_C{}^{n-1} b_C\,]$
  - $(P_{1C})^{-1}$ is structured as

$$(P_{1c})^{-1} = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{n-1} & 1 \\ \alpha_2 & \alpha_3 & \cdots & \cdots & 1 & 0 \\ \alpha_3 & \cdots & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{n-1} & 1 & \cdots & \cdots & 0 & 0 \\ 1 & 0 & \cdots & \cdots & 0 & 0 \end{bmatrix}$$

α_i : coefficients of the monic characteristic polynomial of the matrix A

$$q_A(\lambda) = \lambda^n + \lambda^{n-1}\alpha_{n-1} + \cdots + \alpha_0$$

Controllable Canonical Form

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$P_1 = [B,\ AB,\ A^2B] = \begin{bmatrix} 0 & 2 & 10 \\ 1 & -1 & 3 \\ 1 & 7 & 29 \end{bmatrix}$$

$$det(\lambda I - A) = \lambda^3 - 5\lambda^2 - \lambda + 7$$

$$P_{1c}^{-1} = \begin{bmatrix} \alpha_1 & \alpha_2 & 1 \\ \alpha_2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -5 & 1 \\ -5 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$T = P_1(P_{1c}^{-1}) = \begin{bmatrix} 0 & 2 & 0 \\ 7 & -6 & 1 \\ -7 & 2 & 1 \end{bmatrix}$$

$$T^{-1} = \begin{bmatrix} 0.2857 & 0.0714 & -0.0714 \\ 0.5 & 0 & 0 \\ 1 & 0.5 & 0.5 \end{bmatrix}$$

$$A_c = T^{-1}AT = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -7 & 1 & 5 \end{bmatrix}$$

$$B_c = T^{-1}B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

In controllable canonical form:
Coefficients of the characteristic polynomial

Pole Placement

- ***Property:*** Let us consider the linear stationary system *S = (A, b, C, d)* with one input only and fully reachable. For any *n*-order monic polynomial $p(\lambda)$ a matrix k^T of dimension $1 \times n$ exists such that the characteristic polynomial of the state matrix $A + b\,k^T$ of the feedback system coincides with $p(\lambda)$.

- *Proof:* α_i, $i = 1, \dots, n$ are the coefficients of the characteristic polynomial $q_A(\lambda)$ of the matrix *A*

$$q_A(\lambda) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_0$$

and d_i, $i = 1, \dots, n$ are the coefficients of the desired polynomial $p(\lambda)$

$$p(\lambda) = \lambda^n + d_{n-1}\lambda^{n-1} + \cdots + d_0$$

The couple (A, b) is controllable, therefore a transformation matrix that pose the system in controllable canonical form exists

$$x_c = T^{-1}x, \qquad A_c = T^{-1}AT, \qquad b_c = T^{-1}b$$

Pole Placement

- By adopting the control law:

$$u(t) = k_c^T x_c(t) + v(t) \qquad k_c^T = [k_0,\ k_1,\ \dots,\ k_{n-1}]$$

we obtain:

$$\dot{x}_c(t) \quad = \quad (A_c + b_c k_c^T)\, x_c(t) + b_c\, v(t)$$

with:

$$A_c + b_c k_c^T = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & & & \vdots \\ k_0 - \alpha_0 & k_1 - \alpha_1 & & & k_{n-1} - \alpha_{n-1} \end{bmatrix}$$

whose characteristic polynomial is:

$$q(\lambda) = \lambda^n + (\alpha_{n-1} - k_{n-1})\lambda^{n-1} + \dots + (\alpha_0 - k_0)$$

the desired result is obtained by posing: $\boxed{k_i = \alpha_i - d_i}$

NOTE: $k^T = k_c^T T^{-1} = k_c^T (P_1\ P_{1c}^{-1})^{-1}$

Pole Placement

- *Example:* computing the static state feedback matrix $u(t) = k^T x(t)$ such that the eigenvalues of the closed-loop system are *-1, -2, -2*.

$$\dot{x}(t) = A\,x(t) + b\,u(t) \qquad A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad P_1 = [b \;\; Ab \;\; A^2 b] = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

The system is fully reachable since $rank(P_1) = 3$

The characteristic polynomial of the matrix A and the desired one are respectively

$$p_A(\lambda) = det(\lambda I - A) = det\left(\begin{bmatrix} \lambda - 1 & -2 & 0 \\ 0 & \lambda & -1 \\ 0 & -1 & \lambda \end{bmatrix}\right) = \lambda^3 - \lambda^2 - \lambda + 1$$

$$\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = -1$$

$$p(\lambda) = (\lambda + 2)^2 (\lambda + 1) = \lambda^3 + 5\lambda^2 + 8\lambda + 4$$

It follows

$$\begin{aligned} k_c^T &= [\alpha_0 - d_0, \;\; \alpha_1 - d_1, \;\; \alpha_2 - d_2] \\ &= [-3, \;\; -9, \;\; -6] \end{aligned}$$

Pole Placement

- The vector k^T is obtained from $k^T = k_c^T T^{-1} = k_c^T (P_1 P_{1c}^{-1})^{-1}$

$$P_1 = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad P_{1c}^{-1} = \begin{bmatrix} -1 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

From the charact. Pol. ofA

$$p_A(\lambda) = \lambda^3 - \lambda^2 - \lambda + 1$$

$$P_{1c}^{-1} = \begin{bmatrix} \alpha_1 & \alpha_2 & 1 \\ \alpha_2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} k^T &= [-3, \ -9, \ -6] \left\{ \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right\}^{-1} \\ &= [-3, \ -9, \ -6] \left\{ \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \right\}^{-1} \\ &= [-3, \ -9, \ -6] \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \frac{1}{2} \\ &= [-9, \ -6, \ 3] \end{aligned}$$

Pole Placement

- Verification:

$$A' = (A + bk^T) = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} [-9, \ -6, \ 3]$$

$$= \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} -9 & -6 & 3 \\ 0 & 0 & 0 \\ -9 & -6 & 3 \end{bmatrix}$$

$$\rightarrow \quad p_{A'}(\lambda) = \lambda^3 + 5\lambda^2 + 8\lambda + 4$$

$$= \begin{bmatrix} -8 & -4 & 3 \\ 0 & 0 & 1 \\ -9 & -5 & 3 \end{bmatrix}$$

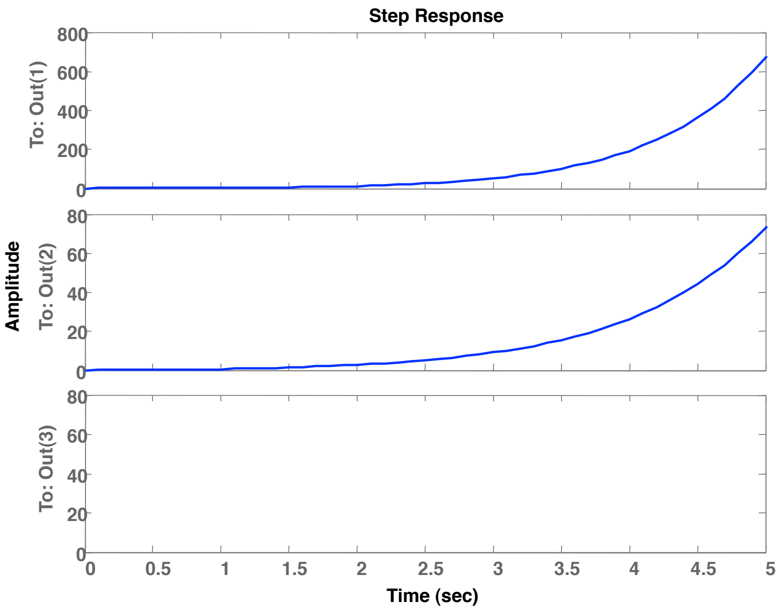
$$A_c = T^{-1}AT = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

$$A'_c = T^{-1}A'T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -8 & -5 \end{bmatrix}$$

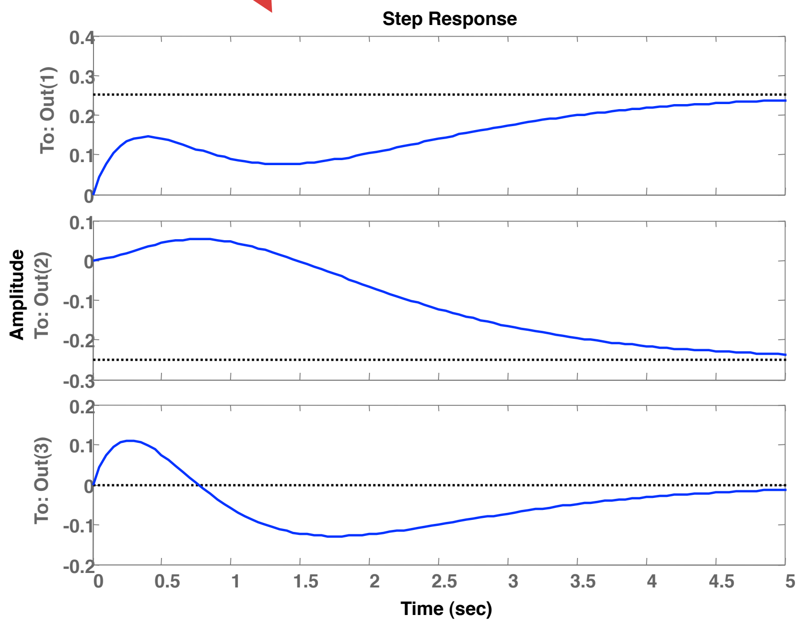
Pole Placement

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$A' = \begin{bmatrix} -8 & -4 & 3 \\ 0 & 0 & 1 \\ -9 & -5 & 3 \end{bmatrix}$$



Unstable state behavior



Stable state behavior

Observable Canonical Form

- ***Property:*** the system $S = (A, c)$ *with one output only* is fully observable iff it is algebrically equivalent to the system $S_o = (A_o, c_o)$ in observable canonical form, that is a system whose matrices A_o and c_o are structured as:

$$A_o = \begin{bmatrix} 0 & 0 & \dots & 0 & -\alpha_0 \\ 1 & 0 & \dots & 0 & -\alpha_1 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \dots & 1 & -\alpha_{n-1} \end{bmatrix}$$

$$c_o = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

where α_i are the coefficients of the monic characteristic polynomial of the matrix A

$$q_A(\lambda) = \lambda^n + \lambda^{n-1}\alpha_{n-1} + \cdots + \alpha_0$$

- **NOTE**: The observable canonical form is obtained from the controllable canonical form of the dual system

Observable Canonical Form

- The observability matrices P_2 and P_{2o} of the original system and of the system in observable canonical form are related by

$$P_{2o} = P_2\, T$$

where T is the transformation matrix $x = T\, x_o$ that pose the original system in observer canonical form.

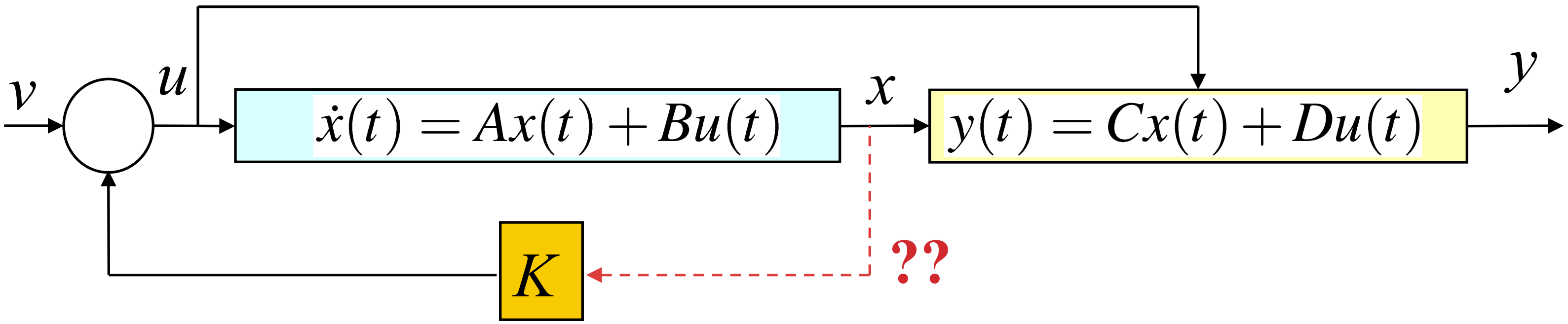
$$P_2\, T = \begin{bmatrix} c \\ cA \\ cA^2 \\ \vdots \\ cA^{n-1} \end{bmatrix} T = \begin{bmatrix} cT \\ cT(T^{-1}AT) \\ cT(T^{-1}A^2T) \\ \vdots \\ cT(T^{-1}A^{n-1}T) \end{bmatrix} = \begin{bmatrix} c_o \\ c_oA_o \\ c_oA_o^2 \\ \vdots \\ c_oA_o^{n-1} \end{bmatrix} = P_{2o}$$

$$\rightarrow \quad T = P_2^{-1}\, P_{2o} \qquad\qquad \rightarrow \quad T^{-1} = P_{2o}^{-1}\, P_2$$

NOTE: $A' = T^{-1}AT \qquad B' = T^{-1}B$

$C' = CT \qquad D' = D$

State Estimator



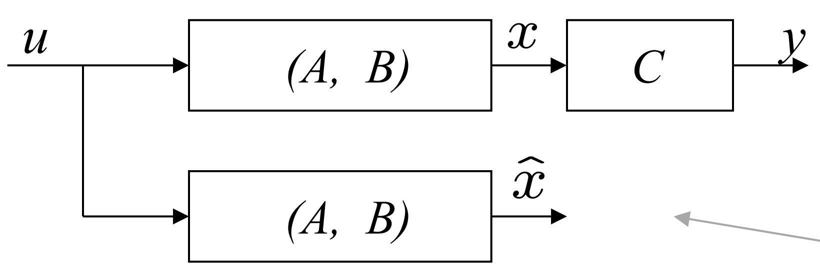
- If the system state is not known, or not directly accessible for the measurement, the design of a "system" that allows to reconstruct the evolution of the state *x(t)* on the basis of the system initial conditions, inputs (and outputs) is needed.

- This "system" is named *state estimator (or observer).*

- Different estimators:
  - *Open-loop:* the estimation is based on the knowledge of the initial cond. and the input
  - *Closed-loop:* the estimation is based also on the knowledge of the output
  - *Reduced-order:* that gives a non-redundant estimation of the state *x(t)*

Open Loop State Estimator

- The open-loop estimator is a copy of the original system:



"Physical" system (real)

Mathematical model (simulated)

- The equations of the open-loop estimator are:

$$\dot{\hat{x}}(t) = A\,\hat{x}(t) + B\,u(t)$$

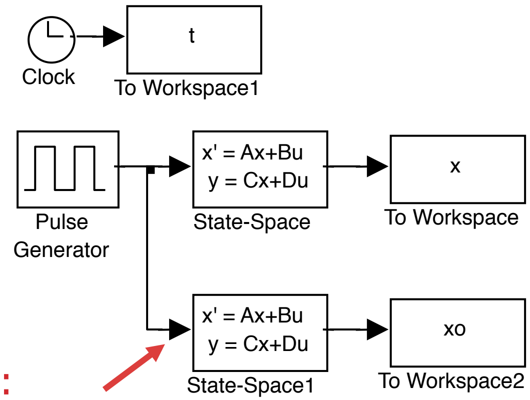
- If the estimation error is defined as $e(t) = x(t) - \hat{x}(t)$

$$\begin{aligned} \dot{e}(t) &= \dot{x}(t) - \dot{\hat{x}}(t) = Ax(t) + Bu(t) - A\hat{x}(t) - Bu(t) \\ &= Ae(t) \end{aligned}$$

$$e(t) = e^{At} e_0$$

The error dynamics depends of the eigenvalues of *A*:
- If the system is unstable, the estimation diverges
- The convergence velocity cannot be changed

Open Loop State Estimator

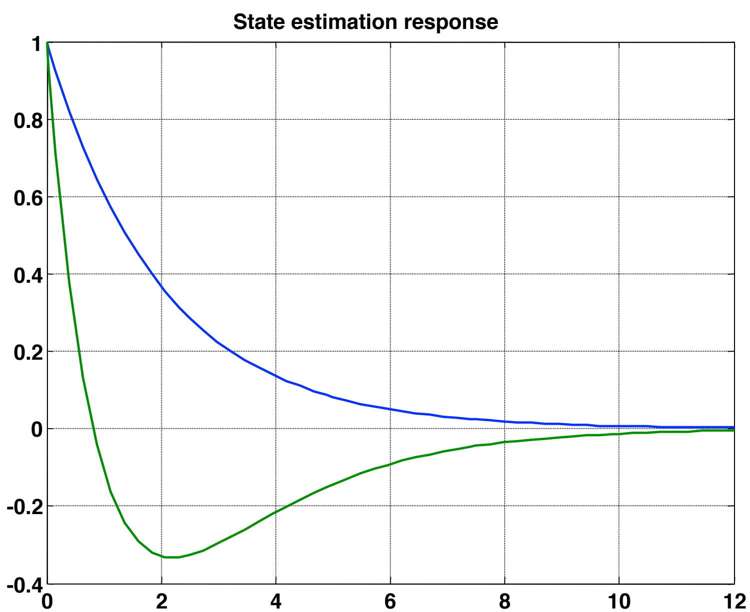


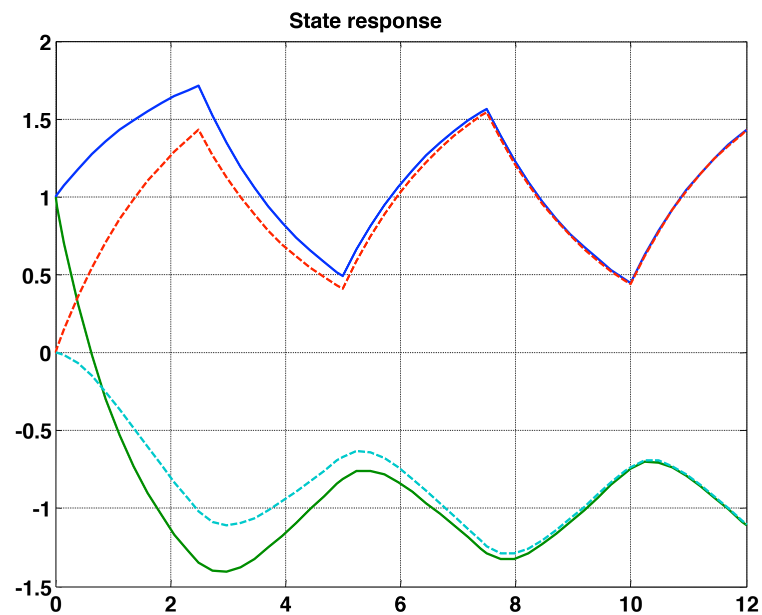
Observer:
a "copy" of the system

$$A = \begin{bmatrix} -0.5 & 0 \\ -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

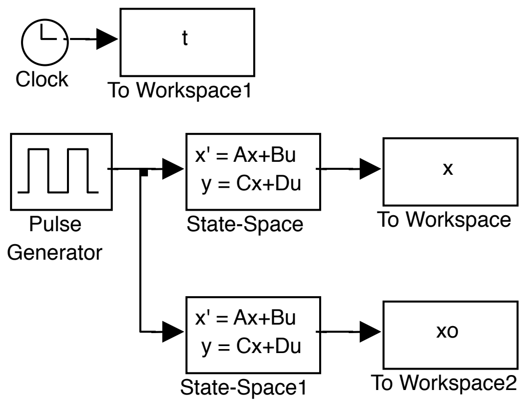
$$x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad x_o(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The initial state
Is unknown



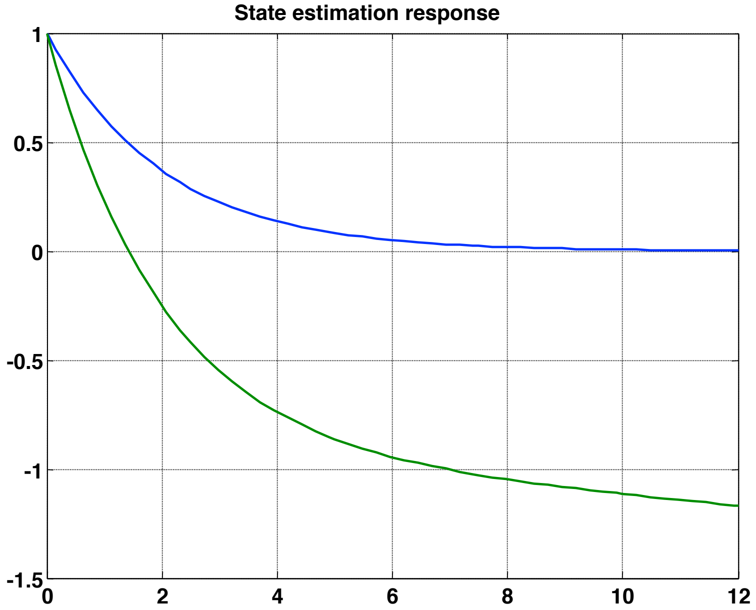


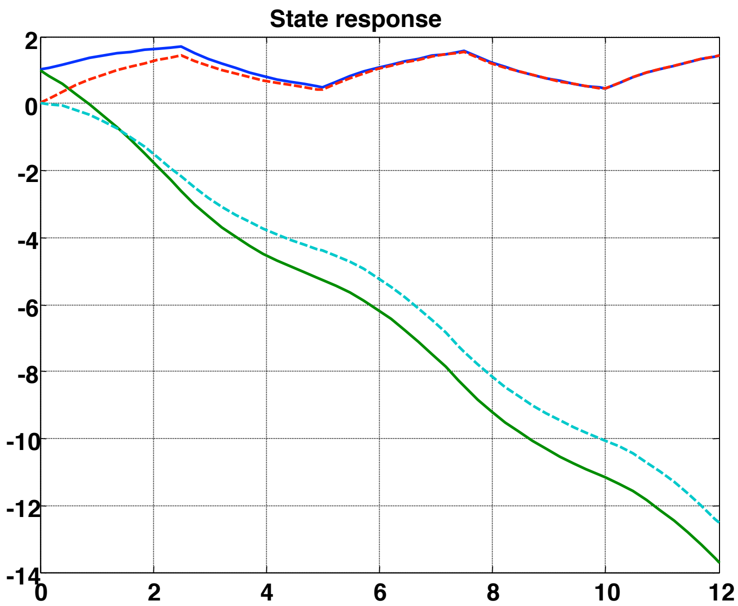
Open Loop State Estimator



$$A = \begin{bmatrix} -0.5 & 0 \\ -1 & 0.02 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad x_o(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



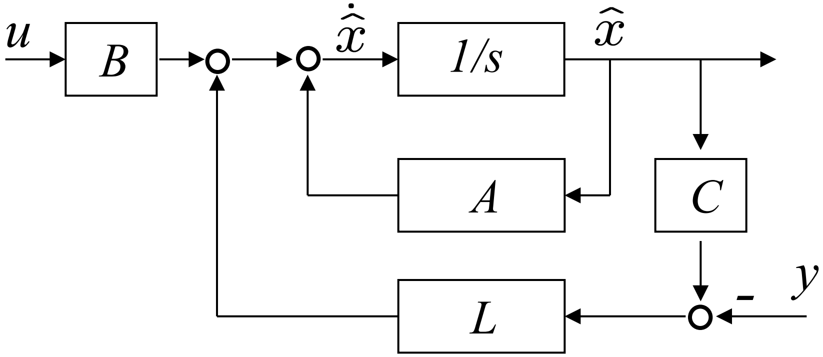


Closed Loop State Estimator

- To solve the two drawbacks previously highlighted, it is possible to adopt a closed-loop estimator, that is named *identity estimator* (its state asymptotically converges to the actual system state)

$$\dot{\hat{x}}(t) = (A + LC)\hat{x}(t) + Bu(t) - Ly(t)$$

where *A, B, C* are (copies of) the matrices of the "observed" system and *L* is a suitably designed matrix.



In this case, the state estimation error dynamics is:

$$\begin{aligned} \dot{e}(t) &= \dot{x}(t) - \dot{\hat{x}}(t) = Ax(t) + Bu(t) - (A + LC)\hat{x}(t) + LCx - Bu(t) \\ &= (A + LC)e(t) \end{aligned}$$

Closed Loop State Estimator

$$\begin{aligned}\dot{e}(t) &= \dot{x}(t)-\dot{\hat{x}}(t) = Ax(t)+Bu(t)-(A+LC)\hat{x}(t)+LCx-Bu(t)\\ &= (A+LC)e(t)\end{aligned}$$

- The state estimation error dynamics is defined by the eigenvalues of the matrix *(A +LC)*

- Thanks to the properties of dual systems, the characteristic polynomial of the matrix *(A+LC)* can be arbitrarily assigned iff the couple (A^T, C^T) is fully reachable, that is *iif the couple (A, C) is fully observable*.

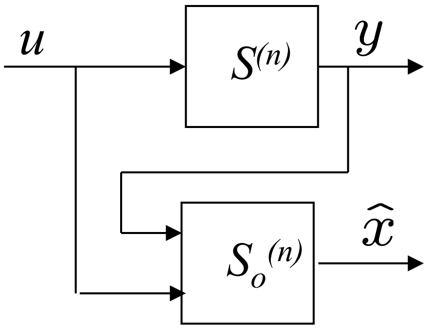
- To compute the matrix *L*, the couple (A^T, C^T) and the techniques for the eigenvalues allocation in the case of fully reachable systems are considered, obtaining in this way the matrix L^T (design by duality with the state feedback problem)

- If the couple *(A, C)* is not fully observable, the state estimation error dynamics characterized by the matrix *(A+LC)* presents some fixed modes that cannot be changed. In this case, the synthesis of an asymptotic state estimator is possible only if the unobservable part of the system is asymptotically stable.

$$L=\left[\begin{array}{c}L_1\\ L_2\end{array}\right],\quad C=[C_1\ \ 0] \quad \Rightarrow \quad (A+LC)=\left[\begin{array}{cc}A_{11}+L_1C_1 & 0\\ A_{21}+L_2C_1 & A_{22}\end{array}\right]$$

Closed Loop State Estimator

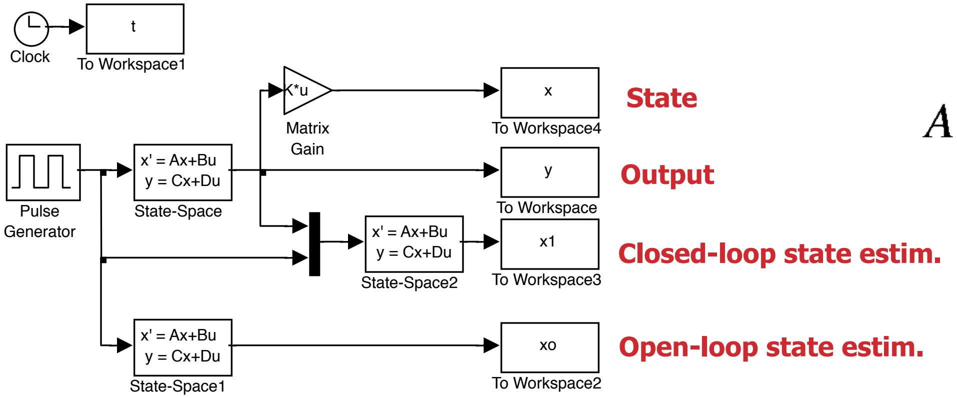
- In the design of the state estimator, given an *n*-order system *S,* a new *n*-order dynamic system S_s is defined (the estimator), obtaining therefore an “overall” *2n*-order system.



- In general, part of the state vector can be directly computed analyzing the output $y = C\,x$, if the matrix C is known.

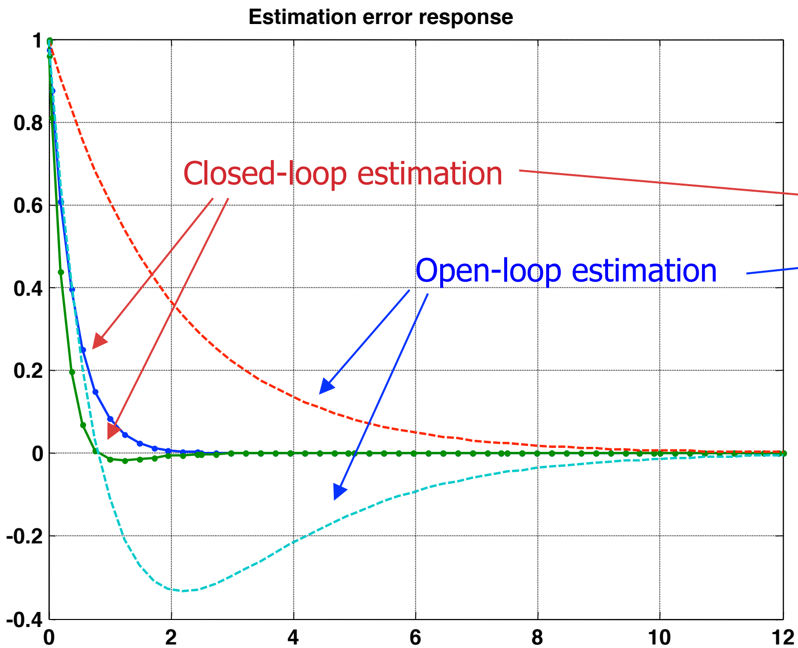
- It is then possible to define a state estimator that reconstructs only the components of the state vector *x* that cannot be directly computed from the output: this state estimator is called *reduced order estimator (or observer)* .

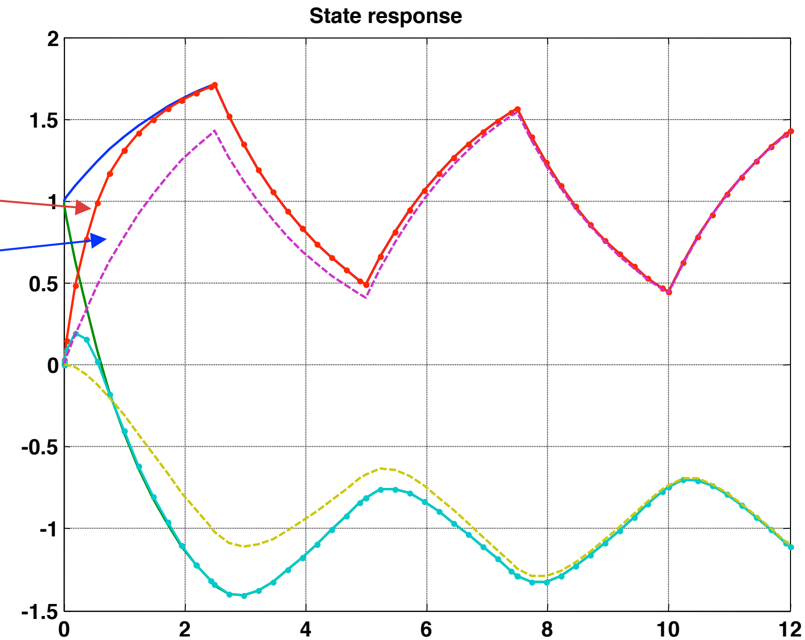
Closed Loop State Estimator



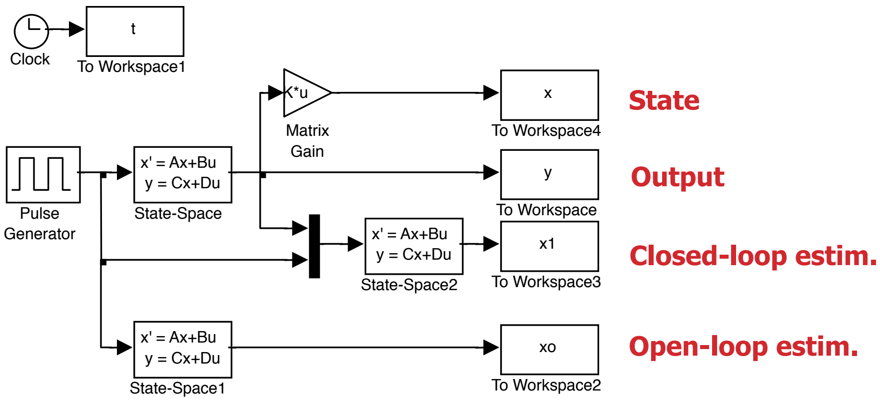
$$A = \begin{bmatrix} -0.5 & 0 \\ -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad x_o(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad L = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$



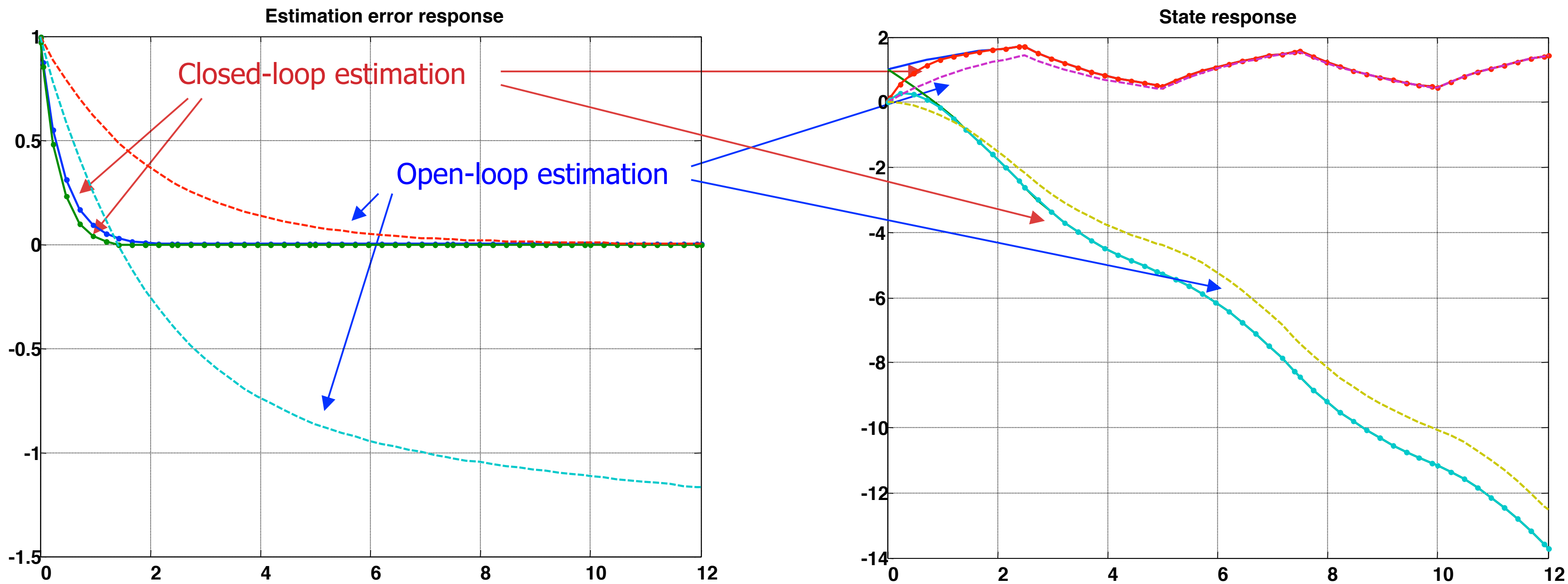


Closed Loop State Estimator



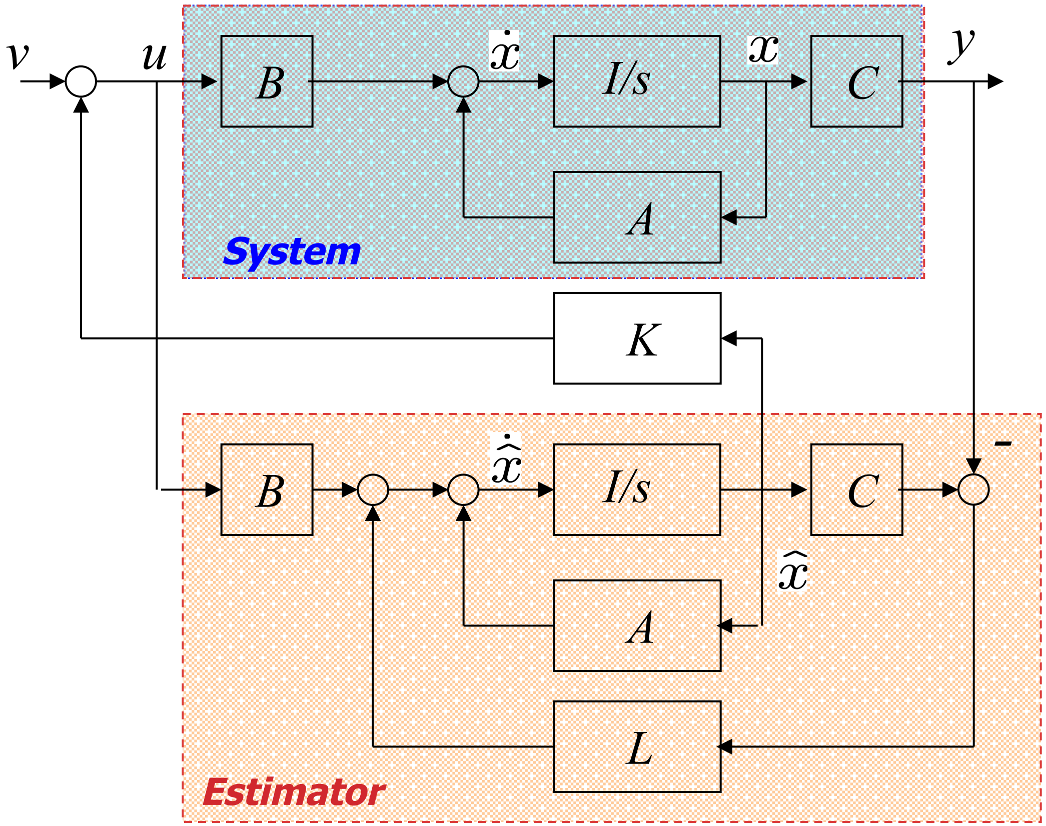
$$A = \begin{bmatrix} -0.5 & 0 \\ -1 & 0.02 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad x_o(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad L = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$



Regulator Design

- We define as *regulator* the system composed by the union of the *state estimator* and of the *static state feedback K*



Regulator Design

- The equations of the overall system are:

$$\begin{aligned}
\dot{x}(t) &= A\,x(t) + B\,u(t) \\
y(t) &= C\,x(t) \\
u(t) &= v(t) + K\hat{x}(t) \\
\dot{\hat{x}}(t) &= A\,\hat{x}(t) + B\,u(t) + L(C\hat{x}(t) - y(t))
\end{aligned}$$

that, by some simple calculation, can be rewritten as:

$$\begin{aligned}
\dot{x}(t) &= A\,x(t) + B\,K\,\hat{x}(t) + B\,v(t) \\
y(t) &= C\,x(t) \\
\dot{\hat{x}}(t) &= (A + L\,C + B\,K)\,\hat{x}(t) - L\,Cx(t) + B\,v(t)
\end{aligned}$$

Regulator Design

- These equations can be grouped in matrix form in a *2n*-order system, defined as:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \end{bmatrix} = \overbrace{\begin{bmatrix} A & B\,K \\ -L\,C & (A+L\,C+B\,K) \end{bmatrix}}^{\bar{A}} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} + \overbrace{\begin{bmatrix} B \\ B \end{bmatrix}}^{\bar{B}} v(t)$$

$$y(t) = \overbrace{\begin{bmatrix} C & 0 \end{bmatrix}}^{\bar{C}} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix}$$

$$\begin{aligned} \dot{\bar{x}}(t) &= \bar{A}\,\bar{x}(t) + \bar{B}\,v(t) \\ y(t) &= \bar{C}\,\bar{x}(t) \end{aligned}$$

- With the aim of highlighting some properties of this system, we apply a state transformation defined as:

$$x' = T\,\bar{x}, \qquad T = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix}, \qquad T^{-1} = T$$

Regulator Design

- It is important to note that the introduced transformation

$$x' = T\bar{x}, \qquad T = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix}, \quad \longrightarrow \quad x' = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} = \begin{bmatrix} x \\ x - \hat{x} \end{bmatrix}$$

generates a state vector *x'* that presents the state of the original system as first component, and the difference between *x* and its estimation (state estimation error) as second component.

- For what is related to the system matrices, we obtain:

$$A' = \begin{bmatrix} (A + B\,K) & -B\,K \\ 0 & (A + L\,C) \end{bmatrix}, \qquad B' = \begin{bmatrix} B \\ 0 \end{bmatrix}$$

$$C' = \begin{bmatrix} C & 0 \end{bmatrix}$$

Regulator Design

- It is important to note that the *presence of the state estimator does not modify the input-output relation of the system*: the transfer function (matrix) of the overall system does not change if the estimated state is considered instead of the actual state *x*.

- From

$$A' = \begin{bmatrix} (A + B\,K) & -B\,K \\ 0 & (A + L\,C) \end{bmatrix}, \qquad B' = \begin{bmatrix} B \\ 0 \end{bmatrix}$$

$$C' = [C \quad 0]$$

it follows

$$\begin{aligned} H(s) &= \bar{C}(sI - \bar{A})^{-1}\bar{B} = C'(sI - A')^{-1}B' \\ &= [C \quad 0] \begin{bmatrix} [sI - (A + B\,K)]^{-1} & \star\star \\ 0 & [sI - (A + L\,C)]^{-1} \end{bmatrix} \begin{bmatrix} B \\ 0 \end{bmatrix} \\ &= C\,[sI - (A + B\,K)]^{-1}\,B \end{aligned}$$

It does not depend on the estimator!

Separation Property

- In the design of the state feedback, only the eigenvalues of the reachable part of the system can be modified ny the relation *(A + BK)*.

- In the same way, it is possible to verify that, if the system state is not directly accessible, a state estimator can be designed in which the dynamics of the observable part can be arbitrarily assigned by suitably selecting the eigenvalues of *(A+LC)*.

- We have also verified that, in case a state estimator in necessary for the regulator implementation, the introduction (and the dynamics) of the state estimator does not affect the design of the state feedback (and viceversa).

Separation Property

- The *separation property* holds. The design of:
  - The state feedback, that is the allocation of the eigenvalues of *(A+BK)*
  - The state estimator, that is the allocation of the eigenvalues of *(A+LC)*

  can be carried out independently.

- The following property holds:

$$
\begin{aligned}
det(sI - A') &= det\left(\begin{bmatrix} (s\,I - A - B\,K) & -B\,K \\ 0 & (s\,I - A - L\,C) \end{bmatrix}\right) \\
&= det(s\,I - A - B\,K)\, det(s\,I - A - L\,C)
\end{aligned}
$$

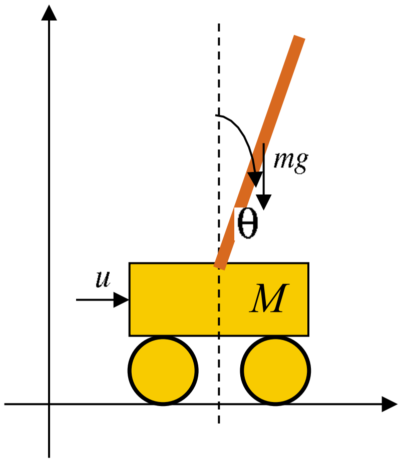
Therefore the eigenvalues of the controller and of the estimator can be independently assigned.

Regulator Design

- *Example:* inverted pendulum position control

  State: angle θ of the pendulum and its velocity

  Output: angle θ



$$\begin{aligned} \dot{x}_1(t) &= x_2(t) = \dot{\theta} \\ \dot{x}_2(t) &= g\frac{ml}{J+ml^2}\,sin x_1 - \frac{ml}{J+ml^2}\frac{1}{M}\,u\,cos x_1 \end{aligned}$$

- By linearizing at the equilibrium point: $x_1 = x_2 = 0$ per $u = 0$

$$\begin{aligned} \dot{x}(t) &= Ax + Bu \\ y(t) &= Cx \end{aligned} \qquad A = \begin{bmatrix} 0 & 1 \\ g\,\alpha & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ -\frac{\alpha}{M} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad \alpha = \frac{ml}{J+ml^2},$$

$$\text{N.B.:} \quad \lambda_{1,2} = \pm\sqrt{g\alpha}\;!$$

$$P_1 = \begin{bmatrix} 0 & -\frac{\alpha}{M} \\ -\frac{\alpha}{M} & 0 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$rank\{P_1\} = 2, \qquad rank\{P_2\} = 2$$

$$K = \begin{bmatrix} k_1 & k_2 \end{bmatrix} \; \textbf{?} \qquad L = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \; \textbf{?}$$

Regulator Design

- Separation property:
  - *Regulator:* since *(A,B)* is fully reachable, the eigenvalues of *(A+BK)* can be arbitrarily assigned

Arbitrary eigenvalues

$$det(sI - A - BK) = s^2 + (\lambda_1 + \lambda_2)s + \lambda_1\lambda_2 = (s+\lambda_1)(s+\lambda_2)$$

$$A + BK = \begin{bmatrix} 0 & 1 \\ g\alpha - \frac{\alpha}{M}k_1 & -\frac{\alpha}{M}k_2 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 \\ -\lambda_1\lambda_2 & -(\lambda_1+\lambda_2) \end{bmatrix}$$

$$\Rightarrow \quad K = \left[(\lambda_1\,\lambda_2 + g\alpha)\frac{M}{\alpha}, \qquad (\lambda_1+\lambda_2)\frac{M}{\alpha}\right]$$

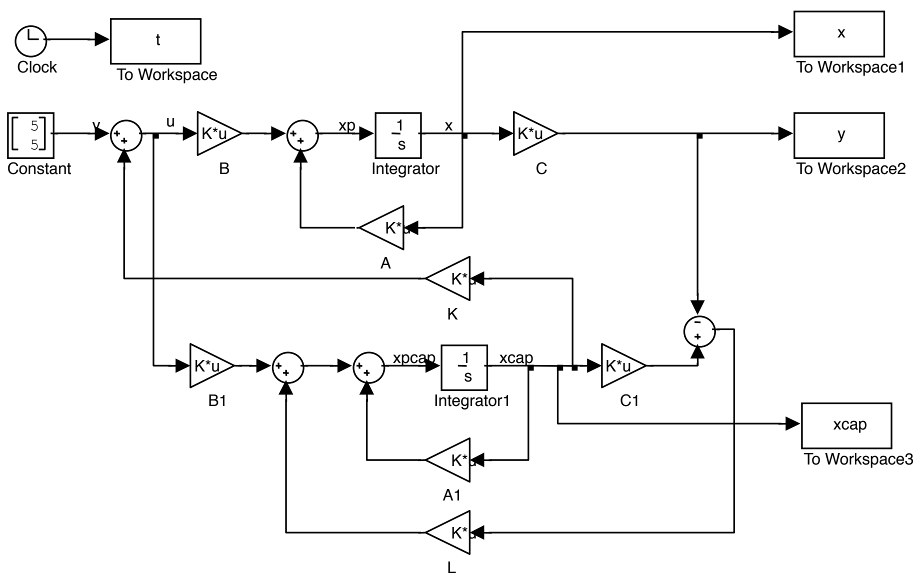
  - *State estimator:* since *(A,C)* is fully observable, even if also x_1 is directly available from the output, the eigenvalues of *(A+LC)* can be arbitrarily assigned

Arbitrary eigenvalues

$$det(sI - A - LC) = (s+\lambda)^2$$

$$A + LC = \begin{bmatrix} l_1 & 1 \\ g\alpha\; l_2 & 0 \end{bmatrix} \Rightarrow s^2 - l_1 s - (g\alpha + l_2) = s^2 + 2\lambda s + \lambda^2 \Rightarrow L = \begin{bmatrix} -2\lambda \\ -\lambda^2 - g\alpha \end{bmatrix}$$

Regulator Design



State

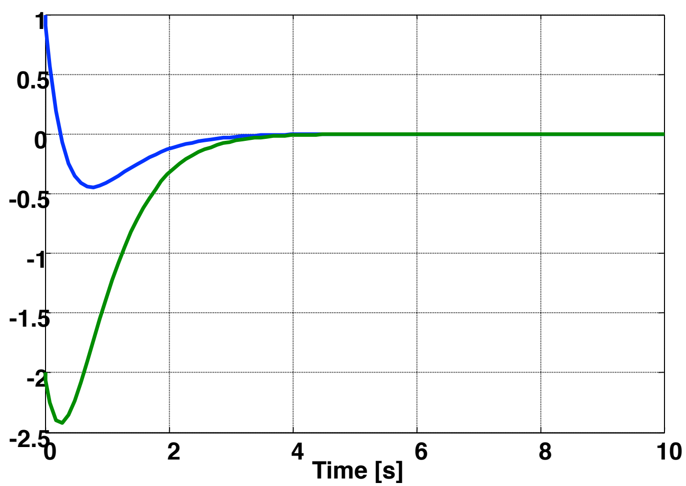
$$\alpha = 5, \quad M = 1$$

$$\lambda_1 = 1, \quad \lambda_2 = 3 \qquad\qquad \lambda = 2$$

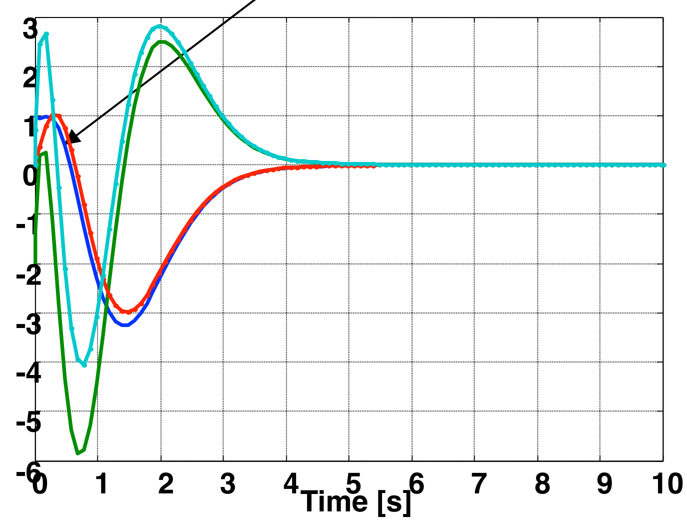
$$K = [10.41, \;\; 0.8] \qquad L = \begin{bmatrix} -4 \\ -53.05 \end{bmatrix}$$

Different initial conditions for the real system and for the state estimator

Estimation error response



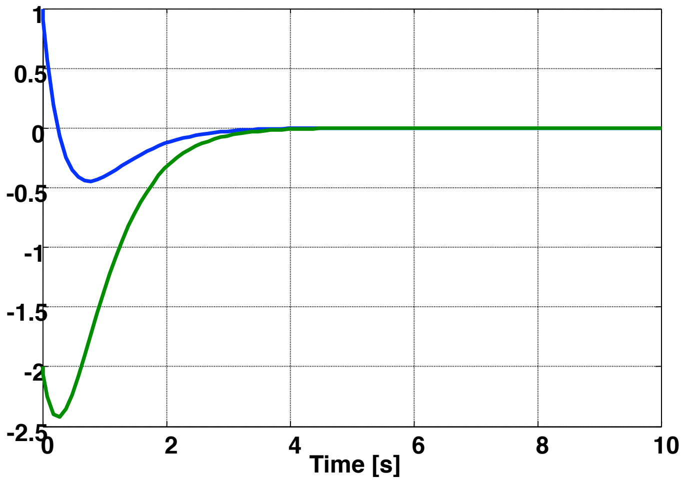
State and Estimation response

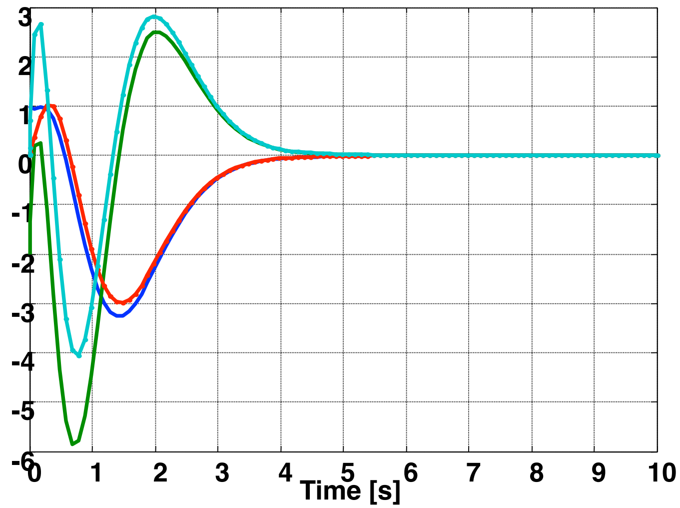


Regulator Design

Estimation error response

State and Estimation response



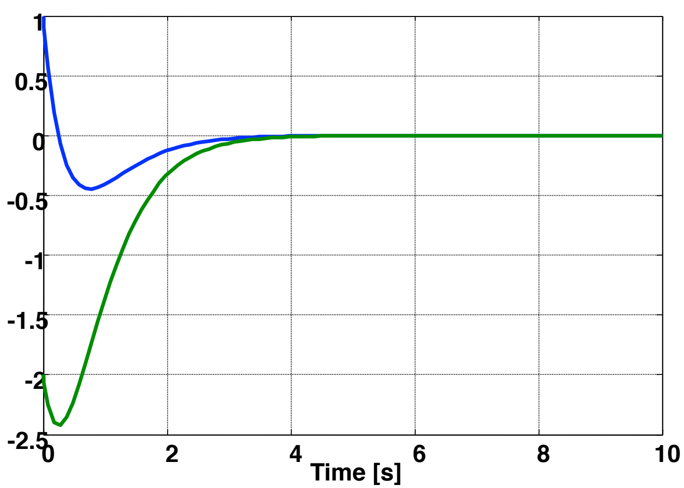


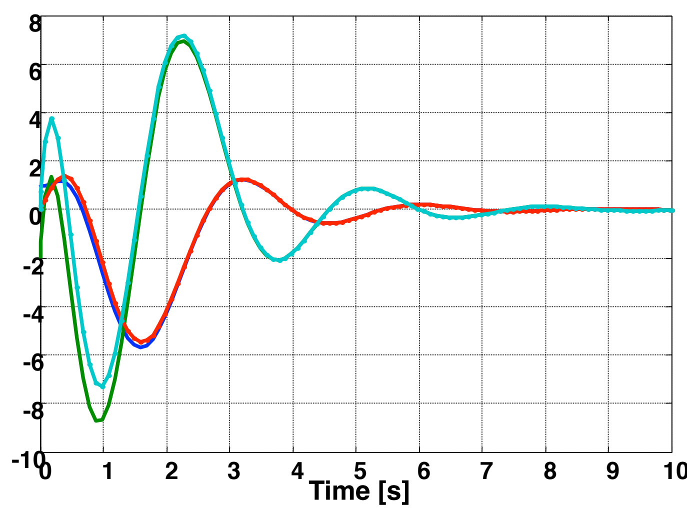
$$\lambda_1 = 1, \quad \lambda_2 = 3$$

$$K = [10.41, \;\; 0.8]$$

$$\lambda = 2$$

$$L = \begin{bmatrix} -4 \\ -53.05 \end{bmatrix}$$





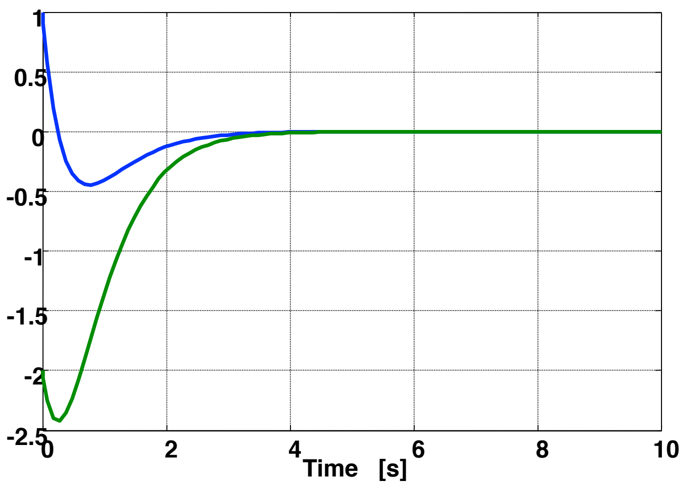
$$\boxed{\begin{gathered} \lambda_1 = 0.1, \quad \lambda_2 = 0.3 \\ K = [9.876, \;\; 0.266] \end{gathered}}$$

$$\lambda = 2$$

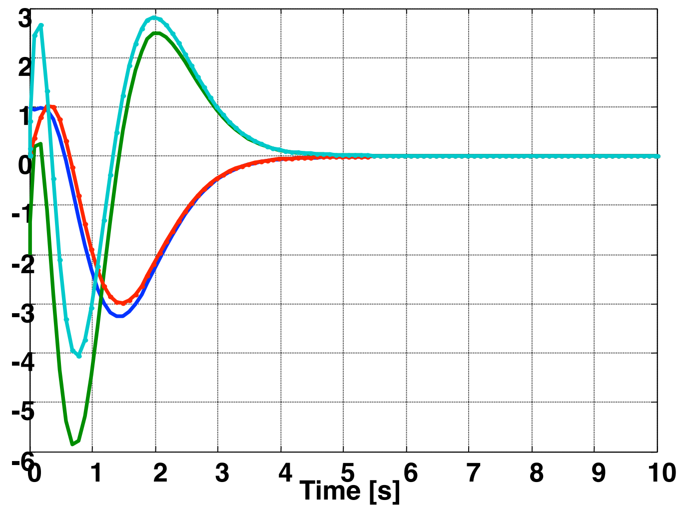
$$L = \begin{bmatrix} -4 \\ -53.05 \end{bmatrix}$$

Regulator Design

Estimation error response



State and Estimation response

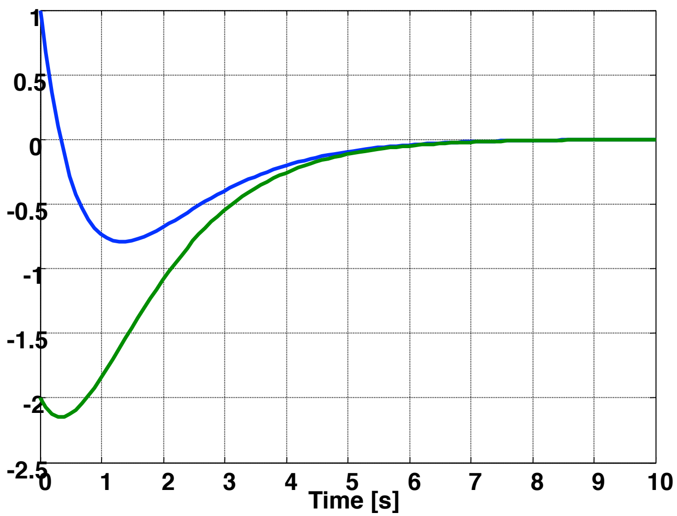


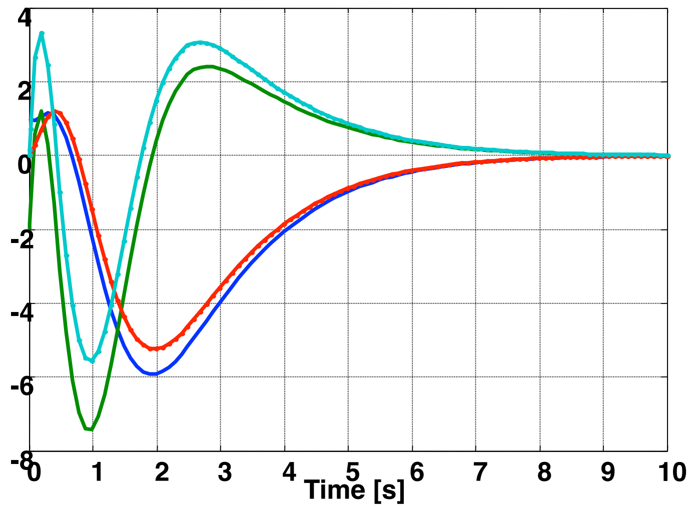
$$\lambda_1 = 1, \quad \lambda_2 = 3$$

$$K = [10.41, \ 0.8]$$

$$\lambda = 2$$

$$L = \begin{bmatrix} -4 \\ -53.05 \end{bmatrix}$$





$$\lambda_1 = 1, \quad \lambda_2 = 3$$

$$K = [10.41, \ 0.8]$$

$$\lambda = 1$$

$$L = \begin{bmatrix} -2 \\ -50.05 \end{bmatrix}$$