AUTOMATIC CONTROL AND SYSTEM THEORY

REDUCED ORDER OBSERVER

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Reduced Order Observer

Problem statement:

Given an *n*-order continuous-time [discrete-time] linear system

$$\begin{aligned} \dot{x}(t) &= Ax(t)+Bu(t) \\ y(t) &= Cx(t)+Du(t) \end{aligned} \qquad \left[\begin{aligned} x(k+1) &= Ax(k)+Bu(k) \\ y(k) &= Cx(k)+Du(k) \end{aligned} \right]$$

with *q* outputs, full rank *C* matrix (*rank(C)=q*) and (*A,C*) fully observable, provide an estimation of the system state by mean of a dynamic system of order (*n-q*).

Solution:

The output information about the *q* components of the state are directly exploited and only the (*n-q*) missing components are estimated.

By means of a state space transformation *T=[T1 T2]* where $T1=C^+$ (right pseudoinverse of *C*) and *ima(T2)=ker(C)*, an equivalent system (*A',B',C',D'*) is obtained such that $C'=[I_q \ \ 0_{(n-q)}]$.

$$A' = T^{-1}AT = \begin{bmatrix} A'_{11} & A'_{12} \\ A'_{21} & A'_{22} \end{bmatrix},\ B' = T^{-1}B = \begin{bmatrix} B'_1 \\ B'_2 \end{bmatrix},\ C' = CT = [I_q \ \ 0_{n-q}],\ D' = D$$

Reduced Order Observer

- **Equivalent system**

Defining as *z* the state of the equivalent system it follows:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} A'_{11} & A'_{12} \\ A'_{21} & A'_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} B'_1 \\ B'_2 \end{bmatrix} u$$

$$y = \begin{bmatrix} I_q & 0_{(n-q)} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + Du = z_1 + Du$$

By means of the change of variable $y_0 = y - D\,u = z_1$ we obtain:

$$\begin{aligned} \dot{z}_1 &= \dot{y}_0 = A'_{11} z_1 + A'_{12} z_2 + B'_1 u = A'_{11} y_0 + A'_{12} z_2 + B'_1 u \\ \dot{z}_2 &= A'_{21} z_1 + A'_{22} z_2 + B'_2 u = \\ &= A'_{21} z_1 + A'_{22} z_2 + B'_2 u + L(-\dot{y}_0 + A'_{11} y_0 + A'_{12} z_2 + B'_1 u) = \\ &= A'_{21} y_0 + (A'_{22} + LA'_{12}) z_2 + B'_2 u + L(-\dot{y}_0 + A'_{11} y_0 + B'_1 u) \end{aligned}$$

where *L* is the $(n\text{-}q)$xq matrix of the reduced-order observer gains.

Reduced Order Observer

Reduced-order observer design

By assuming $w = z_2 + L\, y_0$ we obtain:

$$\begin{aligned} \dot{w} &= \dot{z}_2 + L\dot{y}_0 = \\ &= A'_{21}y_0 + (A'_{22} + LA'_{12})z_2 + B'_2 u + L(A'_{11}y_0 + B'_1 u) = \\ &= (A'_{22} + LA'_{12})w + [-(A'_{22} + LA'_{12})L + LA'_{11} + A'_{21}]y_0 + \\ &\quad + (B'_2 + LB'_1)u \end{aligned}$$

This can be rewritten in more compact form as [dicrete-time case]:

$$\dot{w} = Nw + My_0 + Pu \qquad [w(k+1) = Nw(k) + My_0(k) + Pu(k)]$$

where:

$$\begin{aligned} P &= B'_2 + LB'_1 \\ M &= -(A'_{22} + LA'_{12})L + LA'_{11} + A'_{21} \\ N &= A'_{22} + LA'_{12} \end{aligned}$$

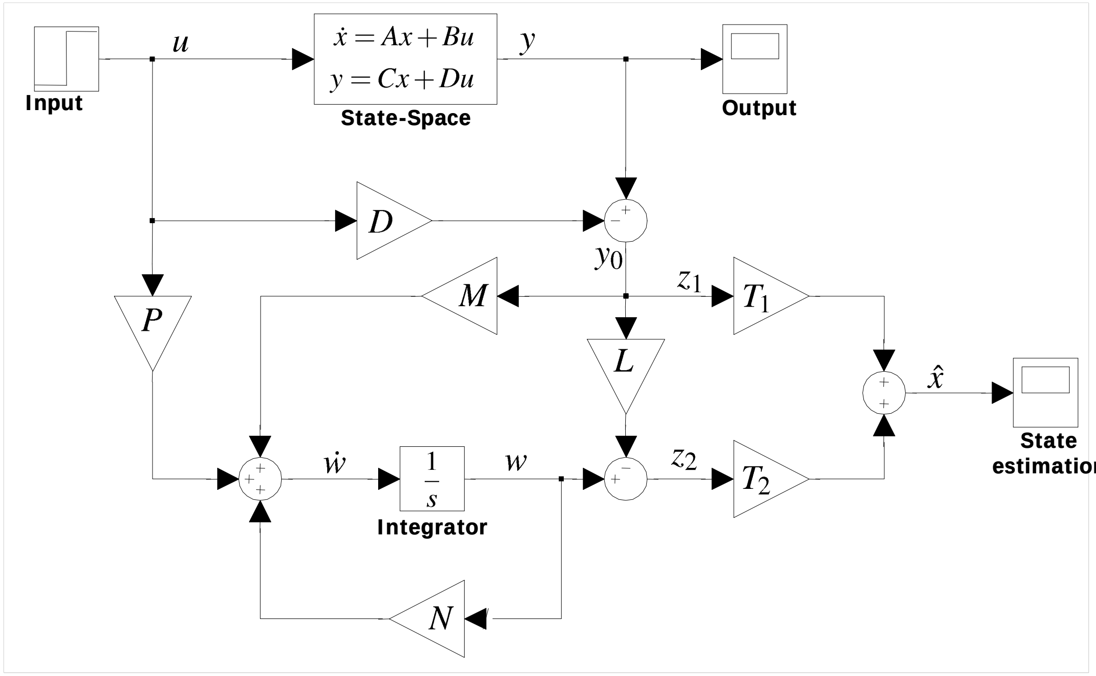
The n-q reduced-order observer eigenvalues can be arbitrarily assigned by means of a suitable choice of the matrix L if the couple (A'_{22}, A'_{12}) is fully observable, this condition is always verified if (A,C) is fully observable and C has rank q.

Reduced Order Observer

Reduced-order observer structure

The reduced-order observer is a (*n*-*q*)-order system that estimates the components of the state that cannot be directly reconstructed from the output.

In this way it is possible to fully exploit the system output and to estimate only the "missing" information about the state.



Separation property

The *2n-q* eigenvalues of the system composed by the static state feedback *K* and by the reduced-order observer are the union (with repetition) of the *n* eigenvalues of $A + B K$ and of the n-q eigenvalues of $A'_{22} + L A'_{12}$.

Reduced Order Observer

- **Separation property for the reduced-order observer (continuous-time case)**

By means of the feedback $u = K\tilde{x}$ and by assuming the error function $e = \tilde{z}_2 - z_2$

$$\dot{e} = \dot{\tilde{z}}_2 - \dot{z}_2 = \dot{w} - L\dot{z}_1 - \dot{z}_2 = Nw + Mz_1 + Pu - [L \;\; I_{(n-q)}]\dot{z}$$

By posing $R = [L \;\; I_{(n-q)}]$ we obtain:

$$\begin{aligned} \dot{e} &= Nw + Mz_1 + Pu - R(A'z + B'u) = \\ &= N(e + z_2 + Lz_1) + Mz_1 + Pu - R(A'z + B'u) \end{aligned}$$

With the given assumption, the following properties hold:

$$P - RB' = 0$$

$$NR + M[I_q \;\; 0_{(n-q)}] - RA' = NR + MC' - RA' = 0$$

The obtained dynamic system with state feedback can be then written as:

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A + BK & BKTS \\ 0 & N \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}, \quad S = [0 \;\; I_{(n-q)}]^T$$