AUTOMATIC CONTROL AND SYSTEM THEORY

PRINCIPLES of OPTIMAL CONTROL

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Control Systems Design

"Classic" techniques – based on:

- Time-domain specifications (rise time, overshoot, steady-state error, …)
- Frequency-domain specifications (phase and/or amplitude margin)

"Iterative" process which (non univocal) solution is often determined by trial and error

Drawbacks:

- Cannot be easily extended to MIMO systems
- The control energy is not explicitely considered
- The "optimal" solution cannot be determined
- Cannot be applied to non-stationary (time-variant) systems
- If the specifications are not satisfied, it is not possible to determine because of the limits of the optimization technique or because specifications incompatibility

Control Systems Design

- *"Modern"* techniques – Based on the definition of a *performance index* (or *objective functional*) that must be minimized, for example:

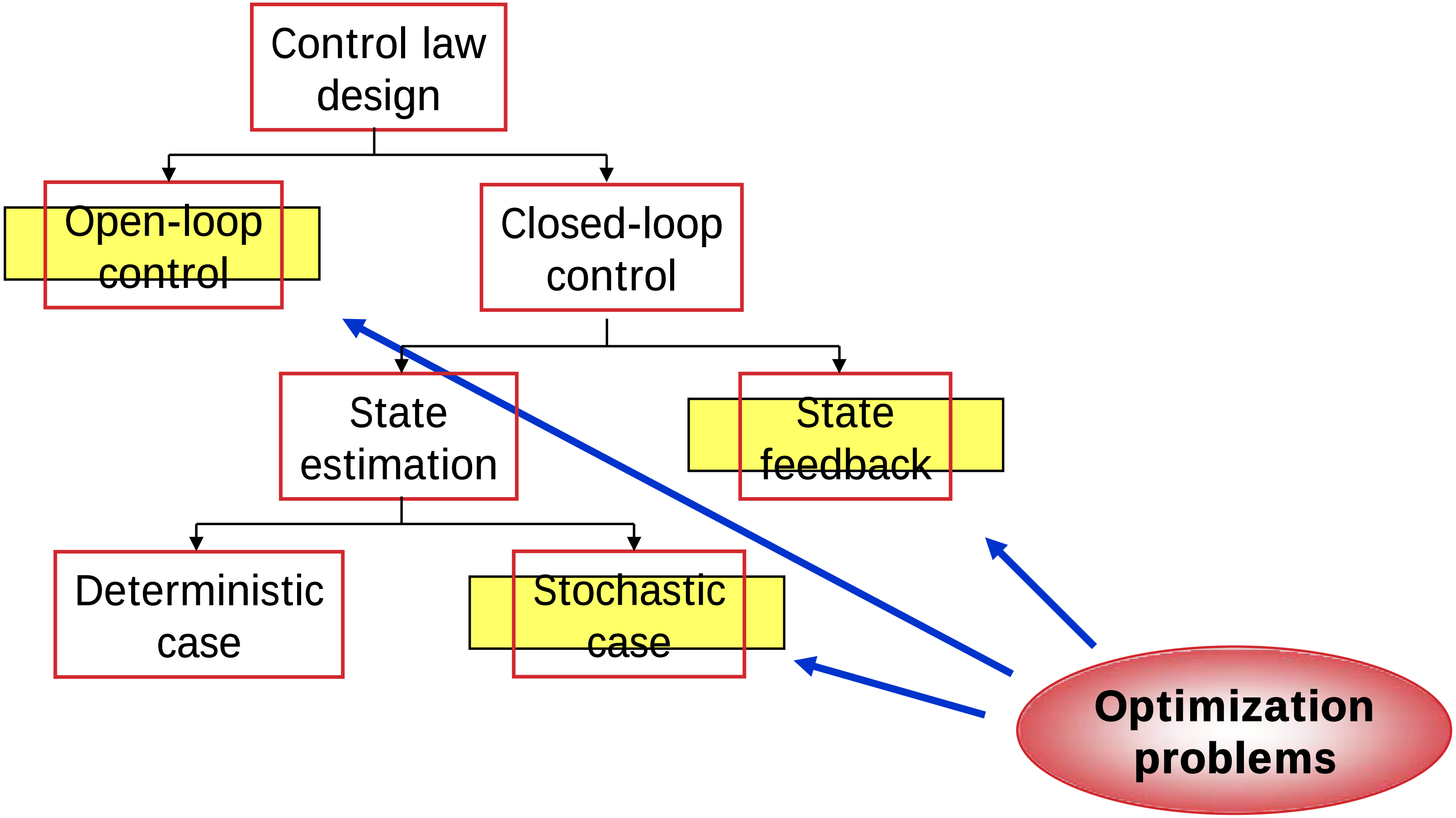
$$PI = \int_0^\infty e^2(t)\, dt$$

- The performance of the system are the defined as *optimal* with respect to the performance index.

- In general, different requirements (sometimes confligting) are considered in the definition of the performance index obtaining in this way *compromise solutions*.

- Similar optimization problems can be found in operative research.

Control Systems Design



Optimal Control

- Given the dynamic system

$$\dot{x}(t) = f(x(t),\, u(t),\, t), \qquad x(t_0) = x_0 \qquad (1)$$

$$\chi[x(t_f), t_f] = 0 \qquad (2)$$

Constraints on the final state

- A general expression for the *performance index* can be:

$$J = \beta[x(t_f), t_f] + \int_{t_0}^{t_f} f_0(x, u, t)\,dt \qquad (3)$$

Cost of the final state

Cost of the state trajectory in [t_0, t_f] or "running cost"

- *Optimal control problem:* computing $u^o(t)$ in $[t_0, t_f]$ that minimize the performance index J.

Optimal Control

$$\begin{aligned} \dot{x}(t) &= f(x(t),\ u(t),\ t), \qquad x(t_0) = x_0 \\ \chi[x(t_f), t_f] &= 0 \end{aligned}$$

- The equation (2) (algebraic and vectorial, with $q \leq n$ relations) is not strictly required in the problem statement, and it represents *the set of admissible states* at the final time t_f.

- A solution exists if at least one of the elements $\chi[x(t_f), t_f]$ is reachable from x_0.
- In general, the final time t_f is free, but some optimization problem can be defined over a desired final time.

Optimal Control

- To derive necessary conditions for the optimum, we will perform the calculus of variations on the cost function (3) subject to the constraints eq. (1) and (2).
- To this end, we define the modified cost function using the Lagrange multipliers $\lambda(t) \in R^n$ *and* $\nu \in R^q$

$$\hat{J} = \beta[x(t_f), t_f] + \nu^T \chi[x(t_f), t_f] + \int_{t_0}^{t_f} [f_0(x,u,t) + \lambda(t)^T (f(x,u,t) - \dot{x})]dt$$

- Let us define the *Hamiltonian* as:

$$H(x,u,\lambda,t) = f_o(x,u,t) + \lambda^T(t) f(x,u,t)$$

Performance index (Lagrangian function)

System dynamics

Costate (or adjoined) variables of dimension *n* (the same as *x*)

Optimal Control

- The variation of $\hat{J}$ is given by assuming independent variations of *u(t), x(t),* λ*(t),* ν and t_f:

$$\delta\hat{J} = \left(\frac{\partial\beta}{\partial x} + \nu^T\frac{\partial\chi}{\partial x}\right)\delta x|_{t_f} + \left(\frac{\partial\beta}{\partial t} + \nu^T\frac{\partial\chi}{\partial t}\right)\delta t|_{t_f} + \chi^T\delta\nu +$$

$$+\int_{t_0}^{t_f}\left[\frac{\partial H}{\partial x}\delta x + \frac{\partial H}{\partial u}\delta u - \lambda^T\delta\dot{x} + \left(\frac{\partial H}{\partial \lambda} - \dot{x}\right)^T\delta\lambda + \frac{\partial H}{\partial t}\delta t|_{t_f}\right]dt$$

- Integrating by parts for $\int_{t_0}^{t_f}\lambda^T\delta\dot{x}\,dt$ yields

$$\delta\hat{J} = \left(\frac{\partial\beta}{\partial x} + \nu^T\frac{\partial\chi}{\partial x} - \lambda^T\right)\delta x(t_f) + \left(\frac{\partial\beta}{\partial t} + \nu^T\frac{\partial\chi}{\partial t} + H\right)\delta t_f + \chi^T\delta\nu +$$

$$+\int_{t_0}^{t_f}\left[\left(\frac{\partial H}{\partial x} + \dot{\lambda}^T\right)\delta x + \frac{\partial H}{\partial u}\delta u + \left(\frac{\partial H}{\partial \lambda}^T - \dot{x}\right)^T\delta\lambda\right]dt$$

Optimal Control

- *An extremum of* $\hat{J}$ *is achieved when* $\delta\hat{J} = 0$ *for all independent variations* δu, δx, $\delta\lambda$, $\delta\nu$ and δt_f.
- It follows that ***necessary condition*** for the problem (1)-(3) to have an optimal solution $u^o(t)$ are

Description	Equation	Variation
Final state constraint	$\chi[x(t_f), t_f] = 0$	$\delta\nu$
State equation	$\dot{x}(t) = \frac{\partial H}{\partial \lambda}^T$	$\delta\lambda$
Costate equation	$\dot{\lambda}(t) = -\frac{\partial H}{\partial x}^T$	δx
Input stationarity	$\frac{\partial H}{\partial u} = 0$	δu
Boundary conditions	$\lambda(t_f) = \left[\left(\frac{\partial \beta}{\partial x} + \nu^T \frac{\partial \chi}{\partial x}\right)^T\right]_{t=t_f}$	$\delta x(t_f)$
	$\left[\frac{\partial \beta}{\partial t} + \nu^T \frac{\partial \chi}{\partial t} + H\right]_{t=t_f} = 0$	δt_f

Optimal Control

- Conditions for the solution to work:
  - The control function must be piecewise continuous (some jumps and/or discountinuities are admissible);
  - The state variables must be continuous and piecewise differentiable;
  - f(.) and f_0(.) first-order differentiable w.r.t. the state variable and t, but not necessarily w.r.t. the control variable u.
  - Finite initial conditions for the state variables are given.
  - If no finite terminal value for state variable at $t=t_f$ is given, then $\lambda(t_0) = 0$.

Optimal Control

- The following are defined *Euler-Lagrange* equations:

$$\dot{\lambda}(t) = -\frac{\partial H}{\partial x}^T, \qquad \lambda(t_f) = \left[\left(\frac{\partial \beta}{\partial x} + \nu^T \frac{\partial \chi}{\partial x}\right)^T\right]_{t=t_f}$$

$$\frac{\partial H}{\partial u} = 0$$

  - The first equation is called *adjoined system*, since it defines the costate (or adjoined variable) dynamics.
  - The second equation is called *stationarity condition*.

- If the final time t_f is not specified, the following relation holds

$$\left[\frac{\partial}{\partial t}(\beta + \nu^T \chi) + H\right]_{t=t_f} = 0$$

- NOTE: *initial conditions* on x ($x(t_0) = x_0$) and *final condizions* on λ ($\lambda(t_f)$) are given, that is in two different time instants (*two-point boundary problem)*. This issue makes the integration of these equations difficult to be solved.

Optimal Control

- In the stationary case the previous condition becomes

$$H\Big|_{t=t_f} = 0$$

- Moreover, from the definition of the Hamiltonian

$$H(x,u,\lambda) \;=\; f_o(x,u) + \lambda^T(t) f(x,u)$$

$$\longrightarrow \quad \frac{dH}{dt} = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial \lambda}\dot{\lambda} + \frac{\partial H}{\partial u}\dot{u}$$

but:

$$\frac{\partial H}{\partial \lambda} = f^T(x,u) = \dot{x}^T, \qquad \dot{\lambda} = -\frac{\partial H}{\partial x}^T, \qquad \frac{\partial H}{\partial u} = 0$$

$\longrightarrow$ $\dfrac{dH}{dt} = 0$ ***H is constant over the optimal trajectory*** $\longrightarrow$ $H = 0$

Optimal Control

There are some interesting cases for the optimal control:

1) No constraint on the final state $x(t_f)$ are given

2) The constraints are independent algebraic functions of the state variables

3) The performance index is given by $T = t_f - t_0$ (minimum-time control)

4) ***"LQ"*** (Linear Quadratic) problem

Optimal Control without Final Constraints

- The set of the admissible state at $t=t_f$ is the whole state space, that is the equation (2) is not present in the problem definition.
- The Euler-Lagrange equations (and the boundery conditions) can be semplified as:

$$\dot{\lambda}(t) = -\frac{\partial H^T}{\partial x}, \qquad \lambda(t_f) = \left[\frac{\partial \beta}{\partial x}^T\right]_{t=t_f}$$

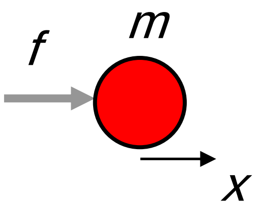
$$\frac{\partial H}{\partial u} = 0$$

- If the final time t_f is not specified, the following relation holds:

$$\left[\frac{\partial \beta}{\partial t} + H\right]_{t=t_f} = 0$$

Optimal Control without Final Constraints

- **Example:** Let us consider a mass *m* in rectilinear motion and a force *f* applied to *m* in the direction of the motion. At the initial time t_0 the position $x(t_0) = x_{10}$ and the velocity x_{20} are known.



- We want to computed the force *f(t)* in $[t_0, t_f]$ with $t_f = 10\ s$, such that at the final time t_f the mass m is:
  - *"Close enough"* to the origin;
  - *"Limiting"* the control action.

$$m\,\ddot{x}(t) = f(t), \qquad x(0) = x_{10}, \quad \dot{x}(0) = x_{20}$$

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), & x_1(0) = x_{10} \\ \dot{x}_2(t) &= u(t), & x_2(0) = x_{20} \end{aligned}$$

with $u(t) = f(t)/m$

Optimal Control without Final Constraints

- On the basis of the specifications, the performance can be defined as:

$$J = c_1\, x_1^2(t_f) + \int_0^{t_f} c_2\, u^2(t)\; dt \qquad\qquad c_1, c_2 > 0$$

where c_1 and c_2 are suitable constants defined to "quantify" the specifications. We assume $c_1 = c_2 = 1$ and $x_{10} = 1\ m$, $x_{20} = 1\ m/s$.

- A possible control law $u(t)$ could be given by solving the differential equation of the system. By imposing $x(t_f) = 0$ and a constant input u, the integration of the differential functions provides:

$$\begin{aligned} x_2(t) &= x_{20} + ut \\ x_1(t) &= x_{10} + x_{20}t + \frac{1}{2}ut^2 \end{aligned} \qquad \rightarrow \qquad x_1(t_f) = 0 = 1 + t_f + \frac{1}{2}u\, t_f^2$$

$$\rightarrow \quad u = -\frac{11}{50} = -0.22 \quad \rightarrow \quad J = c_1\, 0 + \int_0^{10} u^2 dt = 0.484$$

Optimal Control without Final Constraints

- In this case we are not considering the cost of the control action (this control law could be optimal if $c_2 = 0$)

$$J = c_1\, x_1^2(t_f) + \int_0^{t_f} c_2\, u^2(t)\, dt \qquad c_1, c_2 > 0$$

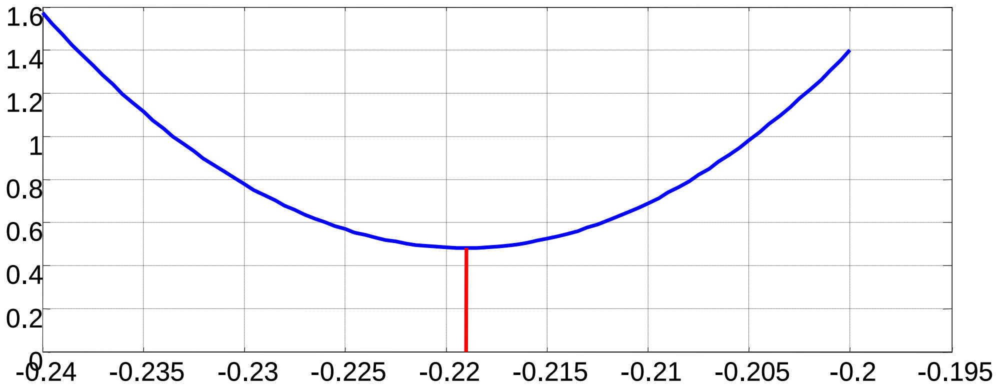
Then, we may suppose that different values of u give a smaller cost J. For example:

$$u = -0.221 \qquad J = 0.4909 \qquad x_{1f} = -0.05, \quad x_{2f} = -1.21$$

$$u = -0.219 \qquad J = 0.4821 \qquad x_{1f} = 0.05, \quad x_{2f} = -1.19$$

$$u = -0.218 \qquad J = 0.4852 \qquad x_{1f} = 0.1, \quad x_{2f} = -1.18$$

$$u = -0.215 \qquad J = 0.5247 \qquad x_{1f} = 0.25, \quad x_{2f} = -1.15$$



Optimal Control without Final Constraints

- From the Hamiltonian function it follows

$$H(x,u,\lambda) = f_o(x,u) + \lambda^T(t) f(x,u)$$

$$\lambda^T = [\lambda_1, \ \lambda_2] \qquad \dot{x} = \begin{bmatrix} x_2 \\ u \end{bmatrix}$$

with $c_1 = c_2 = 1$

$$H(x,u,\lambda) = u^2 + \lambda_1 x_2 + \lambda_2 u$$

and

$$J = x_1^2(t_f) + \int_0^{t_f} u^2(t)\, dt$$

$$\frac{\partial H}{\partial x_1} = 0, \qquad \frac{\partial H}{\partial x_2} = \lambda_1, \qquad \frac{\partial H}{\partial u} = 2u + \lambda_2, \qquad \left.\frac{\partial \beta}{\partial x_1}\right|_{t_f} = 2x_{1f}, \qquad \frac{\partial \beta}{\partial x_2} = 0$$

$$\begin{aligned} \dot{\lambda}_1(t) &= 0, & \lambda_1(t_f) &= 2x_{1f} \\ \dot{\lambda}_2(t) &= -\lambda_1(t), & \lambda_2(t_f) &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \lambda_1(t) &= 2x_{1f} \\ \lambda_2(t) &= -2(t - t_f)x_{1f} \end{aligned}$$

$$u(t) = -\frac{1}{2}\lambda_2(t) \quad \Rightarrow \quad u(t) = (t - t_f)x_{1f}$$

Optimal Control without Final Constraints

- The final state can be computed by integrating the system equations:

$$
\begin{aligned}
x_2(t) &= \frac{1}{2}\, x_{1f}\, t^2 - x_{1f}\, t_f\, t + x_{20} \\
x_1(t) &= \frac{1}{6} x_{1f}\, t^3 - \frac{1}{2} x_{1f}\, t_f\, t^2 + x_{20}\, t + x_{10}
\end{aligned}
$$

$$
\rightarrow \quad x_{1f} = \frac{x_{20}\, t_f + x_{10}}{1 + \frac{1}{3} t_f^3} \approx 0.0329
$$

- Therefore, the optimal control law is $\boxed{u^o(t) = -0.0329(10 - t)}$

It follows that:

$$
x_2(t_f) = -0.645\ m/s, \qquad x_1(t_f) = 0.0329\ m, \qquad \boxed{J = 0.362}
$$

Optimal Control with Final-Time Constraints

- In some cases, the constraints on the final state given by eq. (2) can be expressed as independent constraints on the single state variables (separated constraints). Then, it follows:

$$\begin{aligned} \dot{x}(t) &= f(x(t),\, u(t),\, t), & x(t_0) = x_0 \\ x_i(t_f) &= \chi_i, & i = 1, 2, \ldots, q \leq n \end{aligned}$$

$$J = \beta[x(t_f), t_f] + \int_{t_0}^{t_f} f_0(x, u, t)dt$$

- The boundary conditions for $t = t_f$ in the Euler-Lagrange equations are

$$\dot{\lambda}(t) = -\frac{\partial H^T}{\partial x}, \qquad \lambda(t_f) = \begin{cases} \nu_i, & i = 1, 2, \ldots, q \\ \left[\frac{\partial \beta}{\partial x_i}\right]_{t=t_f} & i = q+1, \ldots, n \end{cases}$$

- ν_i are suitable coefficients to be determined

Optimal Control with Final-Time Constraints

- **Example:** Let us consider again the mass *m* as before, in which the velocity at the final time t_f is assigned $x_2(t_f) = x_{2f}$
- Therefore, the optimal control problem is:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \qquad x_1(0) = x_{10} \\ \dot{x}_2(t) &= u(t), \qquad x_2(0) = x_{20}, \quad x_2(t_f) = x_{2f} \end{aligned}$$

- The performace index is defined as:

t_f **is assigned**

$$J = \int_0^{t_f} u^2(t)\, dt, \qquad t_f > 0$$

$$H(x,u,\lambda) = u^2 + \lambda_1 x_2 + \lambda_2 u$$

$$\frac{\partial H}{\partial x_1} = 0, \quad \frac{\partial H}{\partial x_2} = \lambda_1, \quad \frac{\partial H}{\partial u} = 2u + \lambda_2, \quad \frac{\partial \beta}{\partial x_1} = 0, \quad \frac{\partial \beta}{\partial x_2} = 0$$

Only this term is used

Optimal Control with Final-Time Constraints

- From the Euler-Lagrange equations we obtain:

$$
\begin{aligned}
\dot{\lambda}_1(t) &= 0, & \lambda_1(t_f) = 0 & \qquad \lambda_1(t) = 0 \\
\dot{\lambda}_2(t) &= -\lambda_1(t), & \lambda_2(t_f) = \nu & \qquad \lambda_2(t) = \nu \\
u(t) &= -\frac{1}{2}\lambda_2(t) & & \qquad u(t) = -\frac{1}{2}\nu
\end{aligned}
$$

- By integration of the differential equations

$$
\begin{aligned}
x_2(t) &= -\frac{1}{2}\nu t + x_{20} \\
x_2(t_f) &= x_{2f} = -\frac{\nu}{2}t_f + x_{20} \\
\nu &= \frac{2}{t_f}(x_{20} - x_{2f})
\end{aligned}
$$

$$
\rightarrow \quad u^o(t) = -\frac{x_{20} - x_{2f}}{t_f}
$$

is constant

Minimum-Time Optimal Control

- An optimal control problem is said to be a *minimum time* problem if the performance index is given by the duration of the time interval *[t_0 – t_f]* during which the control is applied.
- The problem can be then written as

$$\dot{x}(t) = f(x(t),\ u(t),\ t), \qquad x(t_0) = x_0 \qquad (1)$$

$$\chi[x(t_f), t_f] = 0 \qquad (2)$$

- and the performance index is simply:

$$J = t_f - t_0 = \int_{t_0}^{t_f} dt$$

- It is important to note that, in general, in this problem the final state is defined (through the constraint eq. (2)) since otherwise the final condition to be reached is not defined and the problem admits the trivial solution $t_f = t_0$, $x_f = x_0$.

Minimum-Time Optimal Control

- The Hamiltonian becomes

$$H(x,u,\lambda,t) \;=\; 1 \;+\; \lambda^T(t)\; f(x,u,t)$$

and the Euler-Lagrange equations

$$\dot{\lambda}(t) \;=\; -\frac{\partial H}{\partial x}^T \;=\; \frac{\partial f}{\partial x}^T \lambda(t), \qquad \lambda(t_f) = \left[\frac{\partial \chi}{\partial x}^T \nu\right]_{t=t_f}$$

$$\frac{\partial H}{\partial u} \;=\; \lambda^T \frac{\partial f}{\partial u} = 0$$

when t_f is not specified, the following condition must be satisfied

$$\left[\nu^T \frac{\partial \chi}{\partial t} + \lambda^T f\right]_{t=t_f} + 1 = 0$$