AUTOMATIC CONTROL AND SYSTEM THEORY

LQ OPTIMAL CONTROL

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LQ Optimal Control

- A particular and very important optimal control problem is defined by the case when:
  - The dynamic system is *linear;*
  - The performance index is defined by *quadratic functions.*

  These problems are called *LQ (Linear – Quadratic) optimal problems.*

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t), \qquad x(t_0) = x_0$$

$$J = x^T(t_f)\,S_f\,x(t_f) + \int_{t_0}^{t_f} [x^T(t)Q(t)x(t) + \underbrace{u^T(t)R(t)u(t)}_{\text{Control Power}}]\,dt$$

$$S_f = S_f^T \geq 0 \qquad Q(t) = Q(t)^T \geq 0 \qquad \underbrace{R(t) = R(t)^T > 0}_{\text{Strictly Pos. def.}}$$

- Usually the matrices S_f, Q *and* R are *diagonal* (the square value of each component of x_f, x and u is considered in the performance index).

LQ Optimal Control

- It is possible to prove that the *necessary conditions* for the solution of the problem are

Hamiltonian $\longrightarrow$ $H = x^T Qx + u^T Ru + \lambda^T (Ax + Bu)$

$$\begin{aligned} \dot{x} &= A(t)x(t) + B(t)u(t), & x(t_0) &= x_0 \\ \dot{\lambda} &= -H_x^T = -2Q(t)x(t) - A^T(t)\lambda(t), & \lambda(t_f) &= 2S_f x(t_f) \\ H_u^T &= 2R(t)u(t) + B^T(t)\lambda(t) = 0 \end{aligned}$$

It follows that $\longrightarrow$ $u(t) = -\frac{1}{2}\underline{R^{-1}}(t)B^T(t)\lambda(t)$

R must be invertible!

LQ Optimal Control – Minimum Energy Control

- A particular and interesting case: $S_f = Q(t) = 0$ e $R(t) = I$.
  If no further conditions are given, the optimal solution is $u(t) = 0$.
  In the stationary case, if the value of the state at the final time $x(t_f)$ and the final time t_f are assigned:

$$\dot{x}(t) \;=\; A\,x(t) + B\,u(t), \qquad x(t_0) = x_0, \qquad x(t_f) = x_f$$

$$J = \int_{t_0}^{t_f} u^T(t)u(t)\,dt$$

  The problem can be seen as finding the minimum-energy control law that drive the state from the initial to the final value. It is here supposed that the system is fully reachable.
- The necessary conditions are:

$$\begin{aligned} \dot{x} &= Ax(t) + Bu(t), & x(t_0) &= x_0 \\ \dot{\lambda} &= -A^T\lambda(t), & \lambda(t_f) &= \lambda_f \\ u(t) &= -\frac{1}{2}B^T\lambda(t) \end{aligned}$$

LQ Optimal Control – Minimum Energy Control

- The costate $\lambda(t)$ must be computed. From the previous equations it follows that:

$$\lambda(t) = e^{-A^T(t-t_f)}\lambda_f$$

therefore

$$u(t) = -\frac{1}{2}B^T(t)e^{-A^T(t-t_f)}\lambda_f$$

- Substituting in the system equations:

$$\begin{aligned}
\dot{x}(t) &= A(t)x(t) - \frac{1}{2}B(t)B^T(t)e^{-A^T(t-t_f)}\lambda_f \\
x(t) &= e^{At}x_0 - \frac{1}{2}\int_0^t e^{A(t-\tau)}BB^T e^{-A^T(\tau-t_f)}\lambda_f d\tau \\
x_f &= e^{At_f}x_0 - \frac{1}{2}\int_0^{t_f} e^{A(t_f-\tau)}BB^T e^{A^T(t_f-\tau)}\lambda_f d\tau \\
&= e^{At_f}x_0 - \frac{1}{2}W_c(t_f)\lambda_f
\end{aligned}$$

LQ Optimal Control – Minimum Energy Control

- We obtain

$$x_f = e^{A\,t_f} x_0 - \frac{1}{2} W_c(t_f) \lambda_f$$

where the matrix W_c is called *controllability gramian* (that is invertible if the system is fully controllable). The value of λ_f can be computed as:

$$\lambda_f = -2[W_c(t_f)]^{-1}(x_f - e^{A\,t_f} x_0)$$

and finally the optimal control law:

$$\boxed{u(t) = B^T e^{-A(t-t_f)} [W_c(t_f)]^{-1}(x_f - e^{A\,t_f} x_0)}$$

Minimum energy control that drives the state from the initial value x_0 to the final value x_f in the time interval $[t_0 - t_f]$.

Optimal Control

- Finite time optimal control problems in *$[t_0 – t_f]$:*
  - ***Open loop:*** the control law does NOT depend from the system state *x(t),* but it is designed on the base of the knowledge of the system model
  - ***Closed loop:*** the control law depends on *x(t)* (state feedback), improving the robustness

- NOTE. The *optimal control law* is unique! If the optimal control is implemented by means of state feedback (closed loop), what we obtain is not a *different control law* (if it exists, it is unique) but a *a different implementation!*

- ***Closed loop:*** the control law is a function of *x(t)* instead of *t*.

  Can this control law be optimal?

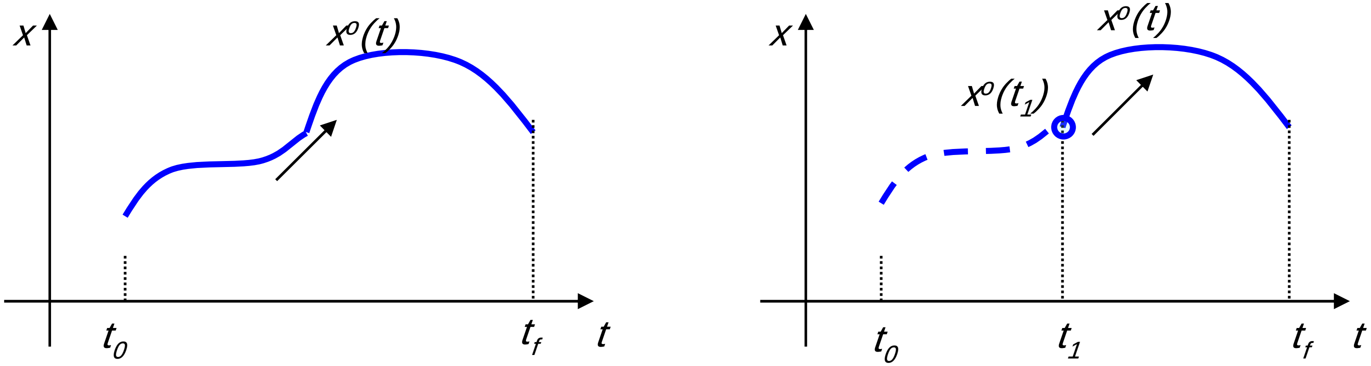
LQ Optimal Control – State feedback formulation

- ***Bellman optimality principle***

  If the control law $u^o(t)$, defined in $[t_0, t_f]$, is optimal w.r.t a given problem with initial conditions (x_0, t_0) and the optimal state trajectory is $x^o(t)$, then the same control law is optimal in $[t, t_f]$ w.r.t. the same problem with initial conditions $[x^o(t), t]$ for each $t \in [t_0, t_f]$.

  It follows that if $u^o(t)$ is the optimal control law in $[t_0, t_f]$, it can be expressede in each time instant t as a function of $x^o(t)$, that is

$$u^o(t) = u^o[x^o(t), t], \qquad \forall t \in [t_0, t_f]$$



- *It is not specified how to express $u^o(t)$ as a fucntion of $x(t)$.*

LQ Optimal Control – State feedback formulation

- LQ optimal control
  - The system to be controlled is *linear*
  - The performance index is composed by *quadratic* functions

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t), \qquad x(t_0) = x_0$$

$$J = x^T(t_f)\,S_f\,x(t_f) + \int_{t_0}^{t_f} \left[x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)\right]\,dt$$

$$S_f = S_f^T \geq 0 \qquad Q(t) = Q(t)^T \geq 0 \qquad R(t) = R(t)^T > 0$$

LQ Optimal Control – State feedback formulation

- Form the general optimal control theory, the Hamiltonian of the system is

$$H = x^T Q x + u^T R u + \lambda^T (Ax + Bu)$$

then, the necessary conditions for the optimal control are:

$$\begin{aligned} \dot{x} &= A(t)x(t) + B(t)u(t), & x(t_0) = x_0 \\ \dot{\lambda} &= -H_x^T = -2Q(t)x(t) - A^T(t)\lambda(t), & \lambda(t_f) = 2S_f x(t_f) \\ H_u^T &= 2R(t)u(t) + B^T(t)\lambda(t) = 0 \end{aligned}$$

It follows that: $u(t) = -\frac{1}{2} R^{-1}(t) B^T(t) \lambda(t)$

$\boxed{\textit{How to compute } \lambda(t) \textit{ ? ?}}$

LQ Optimal Control – State feedback formulation

- For the costate vector $\lambda(t)$ boundary conditions expressed by the relation $\lambda(t_f) = 2 S_f\, x(t_f)$ are given at the final time t_f

- Is it possible to find a linear relation between $\lambda(t)$ and x (t) for each time instant t within the time interval $[t_0, t_f]$?

  In other words we are looking for an expression of λ of the type

$$\lambda(t) = 2S(t)\, x(t), \qquad S(t_f) = S_f$$

- If such a relation exists, then λ depends form the state x and, in the same way, also the control law u(t) is a function of x!

$$\begin{aligned} u(t) &= -\frac{1}{2} R^{-1}(t) B^T(t) \lambda(t) \\ &= -R^{-1}(t) B^T(t) S(t) x(t) \\ &= K(t) x(t) \end{aligned}$$

- **The problem is then how to compute** *S(t)*.

LQ Optimal Control – State feedback formulation

- To compute S(t) we compute the time derivative of

$$\lambda(t) = 2S(t)\, x(t), \qquad S(t_f) = S_f$$

- It results:

$$\begin{aligned}
\dot{\lambda}(t) &= 2\dot{S}(t)\, x(t) + 2S(t)\, \dot{x}(t) \\
&= 2\dot{S}(t)\, x(t) + 2S(t)\, [A(t)x(t) + B(t)u(t)] \\
&= 2\dot{S}(t)\, x(t) + 2S(t)\, [A(t)x(t) - B(t)R^{-1}(t)B^T(t)S(t)x(t)] \\
&= 2\{\dot{S}(t) + S(t)\, [A(t) - B(t)R^{-1}(t)B^T(t)S(t)]\}x(t)
\end{aligned}$$

- On the other hand, from the Euler-Lagrange equations

$$\begin{aligned}
\dot{\lambda}(t) &= -2Q(t)x(t) - A^T(t)\lambda(t) \\
&= -2[Q(t) + A^T(t)S(t)]x(t)
\end{aligned}$$

- By comparing the two relation if follows

$$\dot{S} + S(A - BR^{-1}B^T S) = -Q - A^T S$$

$$\boxed{\dot{S} + SA + A^T S - SBR^{-1}B^T S + Q = 0, \qquad S(t_f) = S_f}$$

LQ Optimal Control – State feedback formulation

- The matrix differential equation

$$\dot{S} + SA + A^T S - SBR^{-1}B^T S + Q = 0, \qquad S(t_f) = S_f$$

is called *differential Riccati equation (DRE)*.

- Since the matrices S_f, Q ed R are symmetric, also $S(t)$ is symmetric

$$\begin{aligned} u(t) &= -\frac{1}{2}R^{-1}(t)B^T(t)\lambda(t) \\ &= -R^{-1}(t)B^T(t)S(t)x(t) \\ &= K(t)x(t) \end{aligned}$$

- The matrix $K(t)$ defines the optimal control law.
- The performance is a trade-off between the state trajectory and the control energy.

LQ Optimal Control – State feedback formulation

- **Example:** We consider (again…) a mass *m* in rectilinear motion with initial velocity x_0 at time t_0. If $x(t)$ is the velocity at time *t*, compute the control law $u(t)$ that minimize the performance index

$$J = c\, x^2(t_f) + \int_{t_0}^{t_f} u^2(t)\, dt$$

- In this case (the position is not considered) we can write the dynamic model of the system as

$$\dot{x}(t) = u(t), \qquad x(t_0) = x_0$$

- From the optimal control theory, the necessary conditions for the *open loop* control law to be optimal are

$$H = f_0(x,u) + \lambda^T f(x,u) = u^2 + \lambda u$$

$$\rightarrow \quad \frac{\partial H}{\partial x} = 0, \qquad \frac{\partial H}{\partial u} = 2u + \lambda, \qquad \left[\frac{\partial \beta}{\partial x}\right]_{t_f} = 2cx_f$$

LQ Optimal Control – State feedback formulation

- From

$$H = f_0(x,u) + \lambda^T f(x,u) = u^2 + \lambda u \qquad \frac{\partial H}{\partial x} = 0, \qquad \frac{\partial H}{\partial u} = 2u + \lambda, \qquad \left[\frac{\partial \beta}{\partial x}\right]_{t_f} = 2cx_f$$

- The Euler-Lagrange equations become

$$\begin{aligned} \dot{\lambda}(t) &= -\frac{\partial H}{\partial x} = 0, \qquad \lambda(t_f) = 2cx_f, && \Longrightarrow \quad \lambda(t) = 2cx_f \\ \frac{\partial H}{\partial u} &= 2u(t) + \lambda(t) = 0 && \Longrightarrow \quad u(t) = -cx_f \end{aligned}$$

- In this case the *optimal control law is constant over time*. For computing the control law x_f is needed. By integrating the state equation we obtain

$$x(t) = x_0 - (t - t_0)cx_f, \qquad x_f = \frac{x_0}{1 + (t_f - t_0)c}$$

$$\boxed{u(t) = -\frac{x_0}{\frac{1}{c} + (t_f - t_0)},} \qquad x(t) = \frac{1 + (t_f - t)c}{1 + (t_f - t_0)c} x_0$$

Optimal control law = constant

LQ Optimal Control – State feedback formulation

- We are now interested in the implementation of a *feedback optimal control*.
- To this end, the DRE must be solved

$$\dot{S} + SA + A^T S - SBR^{-1}B^T S + Q = 0, \qquad S(t_f) = S_f$$

with *A = 0, B = 1, R = 1, Q = 0* (scalar case). We obtain

$$\dot{s}(t) - s^2(t) = 0, \qquad s(t_f) = c$$

$$\frac{ds}{dt} - s^2 = 0, \qquad \frac{ds}{s^2} = dt, \qquad -\frac{1}{s} = t + k, \qquad k = -\frac{1}{s_f} - t_f$$

$$s(t) = \frac{1}{\frac{1}{c} + t_f - t}, \qquad \boxed{u(t) = -\frac{x(t)}{\frac{1}{c} + t_f - t}} \longleftarrow u(t) = -R^{-1}B^T S x(t)$$

The feedback control law SEEMS NOT CONSTANT !!!

LQ Optimal Control – State feedback formulation

- Therefore, two expressions of the optimal control law have been derived:
  The former is a static open-loop control law, the latter is a closed-loop state feedback

$$u(t) = -\frac{x_0}{\frac{1}{c} + (t_f - t_0)}, \qquad u(t) = -\frac{x(t)}{\frac{1}{c} + t_f - t}$$

  On the basis of the Bellmann optimality principle, *the "optimal solution" should be unique…*

- The state evolution computed in the open-loop control law is:

$$u(t) = -\frac{x_0}{\frac{1}{c} + (t_f - t_0)}, \quad \longrightarrow \quad x(t) = \frac{1 + (t_f - t)c}{1 + (t_f - t_0)c} x_0$$

- From the differential equation obtained with the state feedback

$$\dot{x}(t) = u(t) = -\frac{cx(t)}{1 + (t_f - t)c}, \qquad x(t_0) = x_0$$

  the state evolution with the feedback control law is:

$$x(t) = \frac{1 + (t_f - t)c}{1 + (t_f - t_0)c} x_0$$

The state evolution is the same!

LQ Optimal Control – State feedback formulation

- It is possible to show that the cost of the optimal solution can be computed as:

$$J(x_0, t_0) \;=\; x_0^T \, S(t_0) \, x_0$$

  where *S(t)* is the solution of the DRE.
- This result shows that , given the performance index, the cost of the optimal solution *depends on the initial condition only!*
- In other words, since on the basis of the Bellmann optimality principle the optimal solution is unique, the cost depends only from the starting point along the optimal trajectory
- By defining the matrices *F* and *H as*

$$\begin{aligned} \dot{x} &= A(t)x(t) + B(t)u(t) \\ &= A(t)x(t) - B(t)R^{-1}(t)B^T(t)S(t)x(t) \\ &= [A(t) - B(t)R^{-1}(t)B^T(t)S(t)]x(t) \;=\; F(t)x(t) \end{aligned}$$

$$J \;=\; x^T(t_f)\, S_f \, x(t_f) + \int_{t_0}^{t_f} x^T(t)H(t)x(t)\; dt$$

$$H(t) = Q(t) + S(t)B(t)R^{-1}(t)B^T(t)S(t)$$

LQ Optimal Control – State feedback formulation

- The DRE

$$\dot{S} + SA + A^T S - SBR^{-1}B^T S + Q = 0, \qquad S(t_f) = S_f$$

can be written as

Lyapunov-like Equation!

$$\dot{S} + F^T S + SF + H = 0 \qquad S(t_f) = S_f$$

Let us define the Lyapunov-like equation:

$$V(x) = x^T\ S(t)\ x$$

By computing the time derivative

$$\begin{aligned} \dot{V}(x) &= \dot{x}^T S x + x^T S \dot{x} + x^T \dot{S} x \\ &= x^T F^T S x + x^T S F x + x^T[-F^T S - SF - H]x \\ &= -x^T\ H\ x \end{aligned}$$

The performance index becomes

$$\begin{aligned} J &= x^T(t_f)\ S_f\ x(t_f) + \int_{t_0}^{t_f} x^T(t) H(t) x(t)\ dt \\ &= x^T(t_f)\ S_f\ x(t_f) - \int_{t_0}^{t_f} \dot{V}(x) dt \\ &= \cancel{x^T(t_f)\ S_f}\ x(t_f) - x^T\cancel{(t_f)\ S_f}\ x(t_f) + x^T(t_0)\ S(t_0) x(t_0) \end{aligned}$$

LQ Optimal Control – Infinite time horizon

Some Definitions

- Given the linear time-invariant system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \qquad x(t_0) = x_0 \\ y(t) &= Cx(t) \end{aligned}$$

- The system [or the couple (A, B)] is said *stabilizable* if the unstable state subspace is contained in the reachable subspace. It follows that:
  - Every asymptotically stable system is stabilizable
  - Every fully reachable system is stabilizable

- The system [or the couple (A, C)] is said *detectable* if the unobservable subspace is contained in the stable state subspace. It follows that:
  - Every asymptotically stable system is detectable
  - Every fully reconstructable system is detectable

LQ Optimal Control – Infinite time horizon

- Till now, in the defined optimal control problems the final time t_f was
  - assigned
  - finite: $t_f < \infty$.

$\longrightarrow$ Finite time LQ optimal control

- Then, the evolution of the system is defined in t_0 – t_f and, as a consequence it follows that:
  - The matrices that define the optimal control law are time-variant (time–dependent)

$$u(t) = -R^{-1}(t)B^T(t)S(t)x(t) = K(t)x(t)$$

  - The matrix *S(t)* is the solution of the DRE

$$\dot{S} + SA + A^T S - SBR^{-1}B^T S + Q = 0, \qquad S(t_f) = S_f$$

that is often not easy to solve.

LQ Optimal Control – Infinite time horizon

- Given the system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \qquad x(t_0) = x_0 \\ y(t) &= Cx(t) \end{aligned} \tag{1}$$

and the performance index

$$J = x^T(t_f)\, S_f\, x(t_f) + \int_{t_0}^{t_f} \left[x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)\right] dt$$

$$S_f = S_f^T \geq 0 \qquad Q(t) = Q(t)^T \geq 0 \qquad R(t) = R(t)^T > 0$$

NOTE: we can also consider the following performance index

$$\begin{aligned} J &= x^T(t_f)\, S_f\, x(t_f) + \int_{t_0}^{t_f} \left[y^T(t)Q(t)y(t) + u^T(t)R(t)u(t)\right] dt \\ &= x^T(t_f)\, S_f\, x(t_f) + \int_{t_0}^{t_f} \left[x^T(t)C^T Q(t)Cx(t) + u^T(t)R(t)u(t)\right] dt \end{aligned}$$

LQ Optimal Control – Infinite time horizon

- If the couple *(A, B)* is stabilizable and the couple *(A, C)* is detectable, then (*steady-state solution*):

  - The solution *S(t)* of the DRE converges, for $t_f \to \infty$ and for any final condition S_f, to the constant matrix S_∞ given by the equation

$$SA + A^T S - SBR^{-1}B^T S + Q = 0$$

$$(\ SA + A^T S - SBR^{-1}B^T S + C^T QC = 0\ )$$

    that is called *Algebraic Riccati Equation (ARE)*

  - The ARE admits a unique positive semi-definite solution S_∞. If the couple *(A, C)* is fully reconstructable, the matrix S_f is positive definite.

  - The feedback control law

$$u(t) \ = \ -R^{-1}B^T S_\infty \, x(t)$$

    asymptotically stabilizes the system and minimizes the performace index $J(\infty)$ for any S_f (≥ 0).

LQ Optimal Control – Infinite time horizon

Summarizing

- Given the system (1), the *infinite time LQ optimal control problem* consists in defining the optimal control law that minimizes the performance index

$$J = \int_{t_0}^{\infty} [y^T(t)Q(t)y(t) + u^T(t)R(t)u(t)]\, dt$$

$$Q(t) = Q(t)^T > 0 \qquad R(t) = R(t)^T > 0$$

- As shown before, the optimal control law is given by

$$\boxed{u(t) = Kx(t) \qquad K = -R^{-1}B^T S}$$

where the matrix *S* is the unique symmetric positive semi-definite solution of the *ARE*

$$SA + A^T S - SBR^{-1}B^T S + C^T QC = 0$$

- The corresponding value of the performance index is

$$J = x_0^T\, S\, x_0$$

LQ Optimal Control – Infinite time horizon

- Since the matrix S is symmetric, the ARE

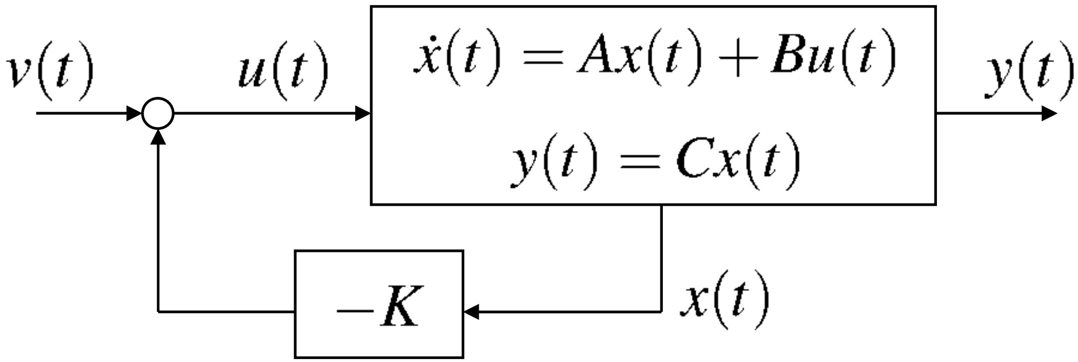
$$SA + A^T S - SBR^{-1}B^T S + C^T QC = 0$$

  gives *n(n+1)/2* scalar algebraic equations.
- Obviously the ARE is simpler to solve than the DRE (finite-time LQ optimal control case).

In Matlab, the instruction *[S, L, K] = care(A,B,Q,R,N)* can be used to compute the state feedback matrix *K* and the matrix *S*. This instruction requires that *R* would be non-singular *R>0* and *N>=0*.

LQ Optimal Control - Robustness

- **Regulator Scheme**



We want to prove that the system shown in the scheme reported above in which the matrix K is computed by means of the ARE with $Q = C^T C$ and $R = I$ presents some useful properties form the point of view of the robustness (amplitude and phase margin).

The following property holds:

Theorem: The feedback regulator shown in the figure, presents in the SISO case an infinite amplitude margin and a phase margin >= 60°.

LQ Optimal Control - Robustness

Lemma: by posing

$$T(s) = C(sI - A)^{-1}B \qquad \text{(input-output t.f.)}$$

$$H(s) = (sI - A)^{-1}B \qquad \text{(input-state t.f.)}$$

where s is an arbitrary parameter (Laplace variable), the following identity holds:

$$(I + H^T(-s)K^T)(I + KH(s)) = I + T^T(-s)T(s)$$

Dim.: Let us consider the ARE with $Q=C^TC$, $R=I$:

$$X\,A + A^T X - X\,B\,B^T X + C^T C = 0$$

Whereas the feedback matrix is:

$$K = B^T X$$

By substituting the expression of the feedback matrix in the ARE:

$$X\,A + A^T X - K^T K + C^T C = 0$$

LQ Optimal Control - Robustness

By summing and subtracting sX

$$-X(sI-A)-(-sI-A^T)X-K^TK+C^TC=0$$

then we pre-multiply by $B^T(-sI-A^T)^{-1}$ and post-multiply by $(sI-A)^{-1}B$.

$$-B^T(-sI-A^T)^{-1}X\cancel{(sI-A)}\cancel{(sI-A)^{-1}}B+$$
$$-B^T\cancel{(-sI-A^T)^{-1}}\cancel{(-sI-A^T)}X(sI-A)^{-1}B+$$
$$-B^T(-sI-A^T)^{-1}(K^TK-C^TC)(sI-A)^{-1}B=0$$

By substituting $XB=K^T$ and $B^TX=K$

$$-B^T(-sI-A^T)^{-1}K^T+K(sI-A)^{-1}B+$$
$$+B^T(-sI-A^T)^{-1}K^TK(sI-A)^{-1}B=$$
$$=B^T(-sI-A^T)^{-1}C^TC(sI-A)^{-1}B=0$$

By summing the identity to both the sides , collecting $B^T(-sI-A^T)^{-1}K^T$ and $K(sI-A)^{-1}B$ and substituting $T(s)$ and $H(s)$ we obtain the desired result.

LQ Optimal Control - Robustness

Now, the harmonic response is considered by posing $s=j\omega$

$$G(j\omega) = K(j\omega I - A)^{-1}B$$ (Internal loop harmonic response)

$$V(j\omega) = C(j\omega I - A)^{-1}B$$ (Open-loop harmonic response)

From the previous results it follows that

$$(I + G^*(J\omega))(I + G(J\omega)) = I + V^*(j\omega)V(j\omega)$$

The matices $G^*(j\omega)G(j\omega)$ and $V^*(j\omega)V(j\omega)$ are hermitian (they are equal to their conjugate traspose) and positive semidefinite. Then the following relation holds:

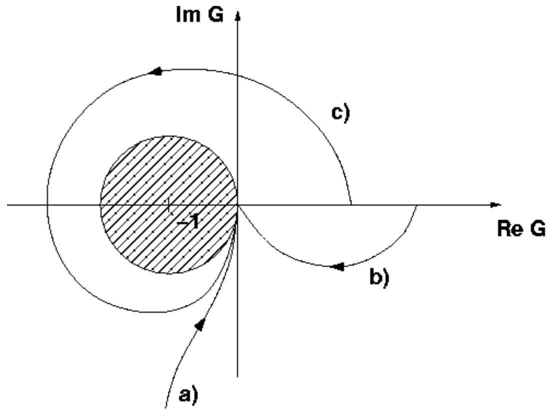
$$(I + G^*(j\omega))(I + G(j\omega)) \geq I$$

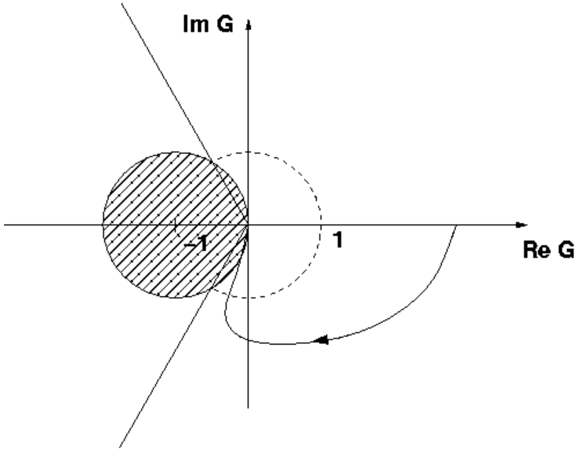
A useful and intuitive intepretation can be given in the SISO case:

$$|1 + G(j\omega)| \geq 1$$

LQ Optimal Control - Robustness

Interpretation in the SISO case





From the previous relation

$$|1+G(j\omega)| \geq 1$$

it can be deduced that the admissible Nyquist plots for this system are of the type reported in these figures: a) and b) refer to stable systems, whereas c) referes to an unstable one.

For what is related to the phase margin, the unit circle meets the unadmissible region in two point that form with the origin angles of 120°, then it follows that the phase margine is >= 60°.

LQG Optimal Control

- LQG optimal control problem
  - The system is *linear*
  - Additive white *Gaussian system noise* *v(t)* and additive *white Gaussian measurement noise* *w(t)* affect the system

$$\begin{aligned} \dot{x}(t) &= A(t)\, x(t) + B(t)\, u(t) + v(t), \qquad x(t_0) = x_0 \\ y(t) &= C(t)x(t) + w(t) \end{aligned}$$

  - The performance index is composed by *quadratic* functions

$$J = E\left(x^T(t_f)\, S_f\, x(t_f) + \int_{t_0}^{t_f} \left[x^T(t)Q(t)x(t) + u^T(t)R(t)u(t) \right] dt \right)$$

$$S_f = S_f^T \geq 0 \qquad Q(t) = Q(t)^T \geq 0 \qquad R(t) = R(t)^T > 0$$

where *E(·)* denotes the *expected value* of its argument.

LQG Optimal Control

- LQG optimal control problem
  - The separation property holds also in the stochastic case if the system is linear
  - Thanks to the separation property, the controller can be designed independently from the observer by solving the LQ optimal control problem in the deterministic case

$$\begin{aligned} \dot{x}(t) &= A(t)\,x(t) + B(t)\,u(t), \qquad x(t_0) = x_0 \\ y(t) &= C(t)x(t) \end{aligned}$$

$$J = x^T(t_f)\,S_f\,x(t_f) + \int_{t_0}^{t_f} \left[x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)\right] dt$$

$$S_f = S_f^T \geq 0 \qquad Q(t) = Q(t)^T \geq 0 \qquad R(t) = R(t)^T > 0$$

$$\begin{aligned} \dot{S}(t) + S(t)A(t) + A^T(t)S(t) - S(t)B(t)R^{-1}(t)B^T(t)S(t) + Q(t) &= 0 \\ S(t_f) &= S_f \end{aligned}$$

The state is unknown!!! $\rightarrow$ $\boxed{u(t) = -R^{-1}(t)B^T(t)S(t)x(t) = K(t)x(t)}$

LQG Optimal Control

- LQG optimal control problem
  - The state of the system is unknown and uncertain
  - A state observer is needed
- General expression for the state estimation

$$\dot{\hat{x}}(t) = A(t)\hat{x}(t) + B(t)u(t) + L(t)(y(t) - C(t)\hat{x}(t)), \quad \hat{x}(0) = E(x(0))$$

$$\begin{aligned} \dot{e}(t) &= \dot{x}(t) - \dot{\hat{x}}(t) = (A(t) + L(t)C(t))e(t) + v(t) \\ y &= C(t)x(t) + w(t) \end{aligned}$$

  - By duality with the optimal control problem, the state estimation could be designed to minimize the performance index

$$J_{\text{obs}} = E\left(x^T(t_f)\, P_f\, x(t_f) + \int_{t_0}^{t_f} \left[x^T(t)V(t)x(t) + y^T(t)W(t)y(t)\right] dt\right)$$

where *V(t), W(t)* and P_f are the covariance matrices of the system noise, the measurement noise and the state at the final time respectively

LQG Optimal Control

- The design of the *optimal observer* can be carried out similarly to the design of the optimal state feedback by solving the DRE

$$\begin{aligned}\dot{P}(t) &= P(t)A(t)^T + A(t)P(t) - P(t)C(t)^T W(t)^{-1} C(t)P(t) + V(t)\\ P(0) &= E(x(0)x(0)^T)\end{aligned}$$

  - NOTE: in this case the DRE is solved forward in time!!!

- The observer feedback matrix is

$$L(t) = P(t)C(t)^T W(t)^{-1}$$

  - This is the so called *Kalman gain matrix*