AUTOMATIC CONTROL AND SYSTEM THEORY

FREQUENCY SHAPING

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Frequency Shaping Control

- ***Parseval Theorem***

  For the signal *u*(*t*), $t \in R$, we can define the (normalized) energy as:

$$\mathcal{E} = \int_{-\infty}^{\infty} |u(t)|^2 dt \quad \left(= \int_{0}^{\infty} |u(t)|^2 dt \right)$$

If the Fourier transform of the signal *u(t),* the computation of the energy can be carried out in the frequency domain:

$$\mathcal{E} = \int_{-\infty}^{\infty} |u(t)|^2 dt \;=\; \int_{-\infty}^{\infty} |U(j\omega)|^2 d\omega$$

Frequency Shaping Control

- Given the system

$$\begin{aligned} \dot{x}(t) &= Ax(t)+Bu(t), \qquad x(t_0) = x_0 \qquad x \in \mathbb{R}^{n\times 1}, \qquad u \in \mathbb{R}^{r\times 1}, \\ y(t) &= Cx(t) \qquad\qquad\qquad\qquad\qquad\quad y \in \mathbb{R}^{m\times 1}, \end{aligned}$$

The methodology for computing the optimal control law is based on the minimization of the performance index

$$J = \int_0^{\infty} [x^T(t)\;\; u^T(t)] \begin{bmatrix} Q & N \\ N^T & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt$$

$$Q = Q^T \geq 0, \quad Q \in \mathbb{R}^{n\times n}, \qquad R = R^T > 0, \quad R \in \mathbb{R}^{r\times r}$$

$$\begin{bmatrix} Q & N \\ N^T & R \end{bmatrix} \geq 0$$

Frequency Shaping Control

- From the definition of the performance index, it is possible to note that the "weights" of x(t) and u(t) is constant for any frequency: The matrcies in the J are constant!

- By means of the Parseval theorem, it is possible to rewrite the performance index as:

$$J = \frac{1}{4\pi} \int_{-\infty}^{\infty} [X^*(j\omega) \;\; U^*(j\omega)] \begin{bmatrix} Q & N \\ N^T & R \end{bmatrix} \begin{bmatrix} X(j\omega) \\ U(j\omega) \end{bmatrix} d\omega$$

  in which X(j ω) and U(j ω) are the Fourier transform of x(t) and u(t). This allows to show that the "weights" in J are constant for any ω.

- In general, we may desire to *design the control system on the basis of some frequency domain specification,* or *to specify different weights for "fast" and "slow" components* of x(t) and u(t).

Frequency Shaping Control

- Then, we are interested in a way to modify the performance index J to take into account for the frequency.

- This desing method is know in literature as *frequency-shaping.*

- Let us consider the performance index

$$J = \frac{1}{4\pi}\int_{-\infty}^{\infty}[X^*(j\omega)\ \ U^*(j\omega)]\begin{bmatrix} Q(j\omega) & 0 \\ 0 & R(j\omega)\end{bmatrix}\begin{bmatrix} X(j\omega) \\ U(j\omega)\end{bmatrix}\, d\omega$$

$$Q(j\omega) = Q^*(j\omega) \geq 0, \qquad R(j\omega) = R^*(j\omega) > 0,$$

- It is possible to assign physically meaningful specifications in the frequency domain if

$$\begin{aligned} Q(j\omega) &= \mathrm{diag}[q_1(\omega^2), \ldots, q_n(\omega^2)] \\ R(j\omega) &= \mathrm{diag}[r_1(\omega^2), \ldots, r_r(\omega^2)] \end{aligned}$$

Frequency Shaping Control

- In this case, the matrices Q and R can be written as

$$Q(j\omega) = P_Q^*(j\omega)P_Q(j\omega), \qquad R(j\omega) = P_R^*(j\omega)P_R(j\omega)$$

where P_R is a r x r matrix, whereas P_Q is a σ x n matrix being in general rank(Q) = $\sigma \leq$ n.

- By defining the new vectors (filtered state and input)

$$X_P(j\omega) = P_Q(j\omega)X(j\omega), \qquad U_P(j\omega) = P_R(j\omega)U(j\omega)$$

the performance index can be written as

$$J = \frac{1}{4\pi}\int_{-\infty}^{\infty} [X_P^*(j\omega) \;\; U_P^*(j\omega)] \begin{bmatrix} X_P(j\omega) \\ U_P(j\omega) \end{bmatrix} d\omega$$

Parseval theorem →

$$J = \int_0^{\infty} [x_P^T(t) \;\; u_P^T(t)] \begin{bmatrix} x_P(t) \\ u_P(t) \end{bmatrix} dt$$

Frequency Shaping Control

- Then we defined an optimal control problem with frequency domain specifications as a “traditional” optimal control problem with infinite time horizon for the variables $x_P(t)$ and $u_P(t)$.

- The problem is now the definition of the new system that takes into account for the frequency domain specifications P_Q e P_R (and then $x_P(t)$ and $u_P(t)$).

- If the elements of $P_Q(j\omega)$ e $P_R(j\omega)$ are proper rational functions, these matrices can be considered as the harmonic response of two linear stationary dynamic system, and their state-space representation is

Order of $P_Q(j\omega)$

$$X_P(j\omega) = P_Q(j\omega)X(j\omega) \rightarrow \begin{array}{rcl} \dot{z}_Q &=& A_Q z_Q + B_Q x \\ x_P &=& C_Q z_Q + D_Q x \end{array} \qquad \begin{array}{l} z_Q \ \eta \times 1 \\ x_P \ \sigma \times 1 \end{array}$$

Order of $P_R(j\omega)$

$$U_P(j\omega) = P_R(j\omega)U(j\omega) \rightarrow \begin{array}{rcl} \dot{z}_R &=& A_R z_R + B_R u \\ u_P &=& C_R z_R + D_R u \end{array} \qquad \begin{array}{l} z_R \ \nu \times 1 \\ u_P \ r \times 1 \end{array}$$

Frequency Shaping Control

- Let us consider the extended system that includes the three dynamic systems with state variables x(t), z_Q(t) and z_R(t).

$$\dot{x}_A(t) = A_A\, x_A(t) + B_A u(t)$$

$$x_A(t) = \begin{bmatrix} x(t) \\ z_Q(t) \\ z_R(t) \end{bmatrix}, \qquad A_A = \begin{bmatrix} A & 0 & 0 \\ B_Q & A_Q & 0 \\ 0 & 0 & A_R \end{bmatrix}, \qquad B_A = \begin{bmatrix} B \\ 0 \\ B_R \end{bmatrix}$$

- Now the preformance index

$$J = \int_0^\infty \left[x_P^T(t) \;\; u_P^T(t)\right] \begin{bmatrix} x_P(t) \\ u_P(t) \end{bmatrix} dt$$

must be rewritten usign x_A(t) and u(t).

Frequency Shaping Control

- Substituting in J the previous state-space representation of x_p and u_p

$$\begin{aligned} x_P^T x_P &= (z_Q^T C_Q^T + x^T D_Q^T)(C_Q z_Q + D_Q x) \\ &= z_Q^T C_Q^T C_Q z_Q + z_Q^T C_Q^T D_Q x + x^T D_Q^T C_Q z_Q + x^T D_Q^T D_Q x \end{aligned}$$

$$\begin{aligned} u_P^T u_P &= (z_R^T C_R^T + u^T D_R^T)(C_R z_R + D_R u) \\ &= z_R^T C_R^T C_R z_R + z_R^T C_R^T D_R u + u^T D_R^T C_R z_R + u^T D_R^T D_R u \end{aligned}$$

$$J = \int_0^\infty [x_A^T(t) \;\; u^T(t)] \begin{bmatrix} Q_A & N \\ N^T & R_A \end{bmatrix} \begin{bmatrix} x_A(t) \\ u(t) \end{bmatrix} \, dt$$

$$Q_A = \begin{bmatrix} D_Q^T D_Q & D_Q^T C_Q & 0 \\ C_Q^T D_Q & C_Q^T C_Q & 0 \\ 0 & 0 & C_R^T C_R \end{bmatrix}, \qquad N = \begin{bmatrix} 0 \\ 0 \\ C_R^T D_R \end{bmatrix}, \qquad R_A = D_R^T D_R$$

Frequency Shaping Control

- Summarizing:

$$\dot{x}_A(t) = A_A\, x_A(t) + B_A u(t)$$

$$x_A(t) = \begin{bmatrix} x(t) \\ z_Q(t) \\ z_R(t) \end{bmatrix}, \qquad A_A = \begin{bmatrix} A & 0 & 0 \\ B_Q & A_Q & 0 \\ 0 & 0 & A_R \end{bmatrix}, \qquad B_A = \begin{bmatrix} B \\ 0 \\ B_R \end{bmatrix}$$

$$J = \int_0^\infty [x_A^T(t) \;\; u^T(t)] \begin{bmatrix} Q_A & N \\ N^T & R_A \end{bmatrix} \begin{bmatrix} x_A(t) \\ u(t) \end{bmatrix} dt$$

Infinite horizon Optimal control

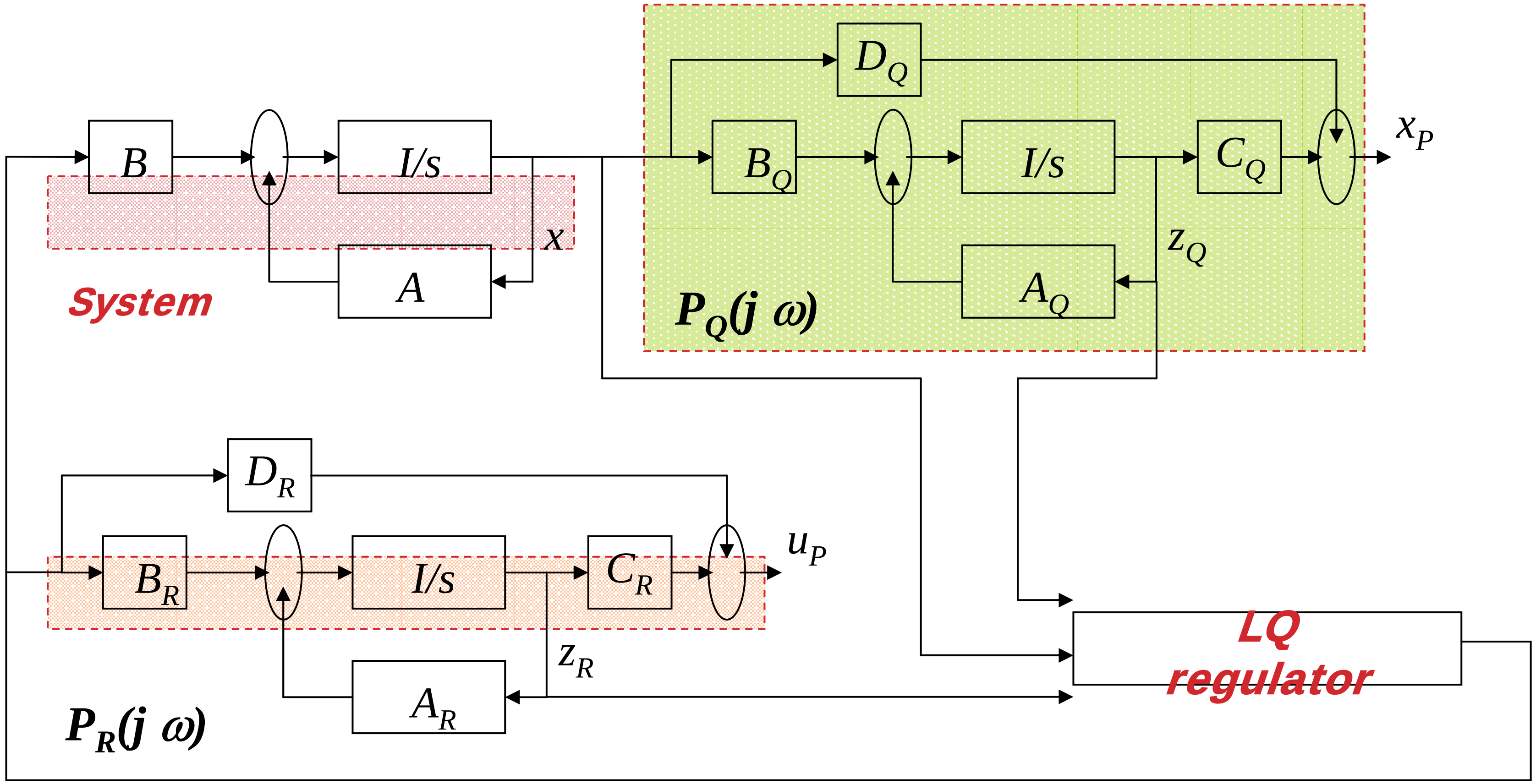
$$u(t) = K\, x_A(t), \qquad K = -R_A^{-1} B_A^T S$$

$$S A_A + A_A^T S - S B_A R_A^{-1} B_A^T S + Q_A = 0, \qquad S \geq 0$$

ARE for the extended system

Frequency Shaping Control

- The overall control scheme is then



It is important to note that the *frequency-shaped optimal control law* consists in a dynamic feedback of the state variables!