Ridig Body Motion – Homogeneous Transformations

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Summary

1. Ridig Body Motion
   - Rotations
   - Translations

2. Homogeneous transformations

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Ridig Body Motion

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Rigid body motions - Homogeneous transformations

Description of the manipulator kinematic properties:

> Description of the geometric characteristics of the robot's motion (position, velocity, acceleration), without considering the forces applied to it

The solution of the kinematic problem is based on:

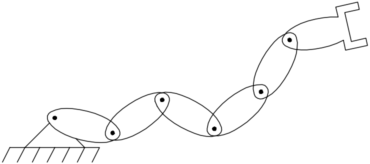
- definition of a reference frame associated to each link of the manipulator
- a procedure for the computation of the relative motion (position, velocity, acceleration) of these frames due to joints' movements.

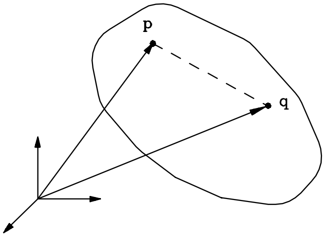
It is necessary to introduce some conventions to describe the position/orientation of rigid bodies and their motion in the space:

- kinematic properties of a rigid body and how to describe them
- homogenous transformations
- description of position and velocity (force) vectors in different reference frames.

Rigid body and its representation

A manipulator is composed by a series of rigid bodies, the links, connected by joints that allow a relative motion.





RIGID BODY: idealization of a solid body of finite size in which deformation is neglected: the distance between any two given points of a rigid body remains constant in time regardless of external forces exerted on it.

$$||\mathbf{p}(t) - \mathbf{q}(t)|| = d(\mathbf{p}(t), \mathbf{q}(t)) = cost$$

Rigid body and its representation

Some assumptions:

- The 3D operational space is represented by the vector space $\mathbb{R}^3$,
- In the 3D space, are defined the inner product

$$\mathbf{u}^T \mathbf{v} = \sum_{i=1}^{n} u_i v_i = ||\mathbf{u}||\,||\mathbf{v}|| \cos\theta \qquad \mathbf{u}, \mathbf{v} \in \mathbb{R}^n$$

  and the Euclidean norm

$$||\mathbf{u}|| = \mathbf{u}^T \mathbf{u} = \sum_{i=1}^{n} u_i^2 \qquad \mathbf{u} \in \mathbb{R}^n$$

- We often use Cartesian (right-handed) reference frames, with homogenous dimensions along the axes
- The base frame is an inertial frame.

Rigid body and its representation

General definition of norm:

$$\|\mathbf{u}\| = \mathbf{u}^T \mathbf{W} \ \mathbf{u}$$

being $\mathbf{W}$ a matrix:

- symmetric
- positive definite

Often, $\mathbf{W}$ is a diagonal matrix.

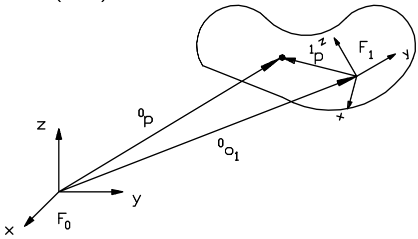
Since $\mathbf{W} = \mathbf{V}^T \mathbf{V}$, then:

$$\|\mathbf{u}\| = \mathbf{u}^T \mathbf{W} \ \mathbf{u} = (\mathbf{u}^T \mathbf{V}^T)(\mathbf{V}\mathbf{u}) = \mathbf{x}^T \mathbf{x}$$

Rigid body and its representation

In $\mathbb{R}^3$, a rigid body has 6 degrees of freedom (dof):

- 3 for the position, x, y, z;
- 3 for the orientation, α, β, γ.



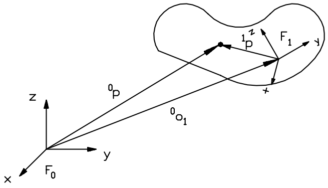
In general, a rigid body in $\mathbb{R}^n$ has

- n dof for the position
- $n(n-1)/2$ dof for the orientation

Rigid body and its representation

Roto-translation motion: most general motion of a rigid body in the space, composed by a rotation about an axis (*instantaneous axis of rotation*) and a translation along the same axis.

Problem of describing the instantaneous position/orientation of a rigid body with respect to a fixed base frame



Let ${}^0\mathbf{o}_1$ be the origin of the frame $\mathcal{F}_1$ fixed to the rigid body, expressed in $\mathcal{F}_0$. Each point of the body has coordinates

$$ {}^1\mathbf{p} = [{}^1p_x \; {}^1p_y \; {}^1p_z]^T $$

constant with respect to $\mathcal{F}_1$.

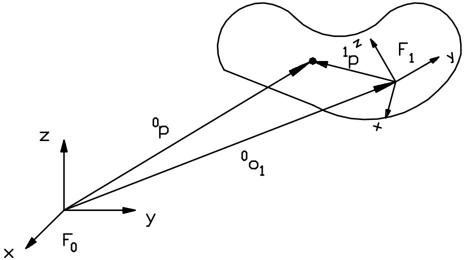
Since the body moves, the same point has coordinates ${}^0\mathbf{p}$, expressed in $\mathcal{F}_0$, variable in time

$$ {}^0\mathbf{p} = [{}^0p_x \; {}^0p_y \; {}^0p_z]^T $$

Rigid body and its representation

First problem: if point **p** is known in $\mathcal{F}_1$, compute the equivalent representation in $\mathcal{F}_0$.

$$^1\mathbf{p} \quad \Longrightarrow \quad ^0\mathbf{p}$$



The problem is solved by using the **Homogeneous Transformation Matrix** $^0\mathbf{T}_1$:

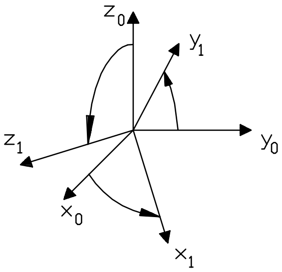
$$^0\mathbf{T}_1 = \left[\begin{array}{ccc|c} & ^0\mathbf{R}_1 & & ^0\mathbf{o}_1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} n_x & s_x & a_x & o_x \\ n_y & s_y & a_y & o_y \\ n_z & s_z & a_z & o_z \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

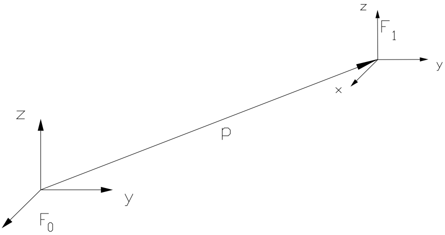
defining the transformation (roto-translation) between $\mathcal{F}_1$ and $\mathcal{F}_0$.

Rigid body and its representation

The problem is decomposed into two parts:

1. $\mathcal{F}_0$ and $\mathcal{F}_1$ share the same origin, and have a different orientation in space
2. $\mathcal{F}_0$ and $\mathcal{F}_1$ have parallel axes but a different origin (translation).





Ridig Body Motion – Homogeneous Transformations

Rotations

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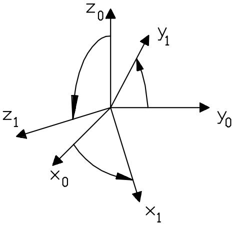
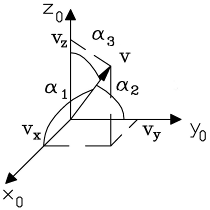
Rotations

Consider two reference frames $\mathcal{F}_0$ and $\mathcal{F}_1$ with the same origin, i.e. $\mathbf{o}_1 \equiv \mathbf{o}_0$. Given a vector ${}^0\mathbf{v}$ in $\mathcal{F}_0$, its components $v_x,\ v_y,\ v_z$ are the orthogonal projections of ${}^0\mathbf{v}$ on the coordinate axes.





$$
\begin{aligned}
v_x &= {}^0\mathbf{v}^T\mathbf{i} = ||{}^0\mathbf{v}||\cos\alpha_1 \\
v_y &= {}^0\mathbf{v}^T\mathbf{j} = ||{}^0\mathbf{v}||\cos\alpha_2 \\
v_z &= {}^0\mathbf{v}^T\mathbf{k} = ||{}^0\mathbf{v}||\cos\alpha_3
\end{aligned}
$$

$\mathbf{i},\ \mathbf{j},\ \mathbf{k}$: unit vectors defining the directions of $x_0,\ y_0,\ z_0$

Rotations

If ${}^0\mathbf{v}$ indicates an axis of $\mathcal{F}_1$, e.g. ${}^0\mathbf{i}_1$, then ${}^0\mathbf{i}_1 = [{}^0i_x \; {}^0i_j \; {}^0i_z]^T$, where

$$
\begin{array}{rcccl}
{}^0i_x & = & {}^0\mathbf{i}_1^T\,\mathbf{i} & = & \cos\alpha_1 \\
{}^0i_y & = & {}^0\mathbf{i}_1^T\,\mathbf{j} & = & \cos\alpha_2 \\
{}^0i_z & = & {}^0\mathbf{i}_1^T\,\mathbf{k} & = & \cos\alpha_3
\end{array}
$$

This is a well known result:

$$
{}^0\mathbf{i}_1 = [\cos\alpha_1,\ \cos\alpha_2,\ \cos\alpha_3]^T
$$

the components of a unit vector with respect to a reference frame are its direction cosines.

A similar results holds for the other directions ${}^0\mathbf{j}_1$ and ${}^0\mathbf{k}_1$.

Rotations

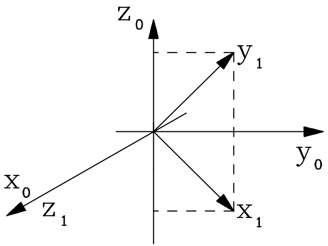
Once the direction cosines of the three axes of $\mathcal{F}_1$ with respect to $\mathcal{F}_0$ are known, the matrix $\mathbf{R}$ may be defined:

$$\mathbf{R} = \begin{bmatrix} {}^0\mathbf{i}_1^T\mathbf{i} & {}^0\mathbf{j}_1^T\mathbf{i} & {}^0\mathbf{k}_1^T\mathbf{i} \\ {}^0\mathbf{i}_1^T\mathbf{j} & {}^0\mathbf{j}_1^T\mathbf{j} & {}^0\mathbf{k}_1^T\mathbf{j} \\ {}^0\mathbf{i}_1^T\mathbf{k} & {}^0\mathbf{j}_1^T\mathbf{k} & {}^0\mathbf{k}_1^T\mathbf{k} \end{bmatrix}$$

- ${}^0\mathbf{i}_1$, ${}^0\mathbf{j}_1$, ${}^0\mathbf{k}_1$: axes of $\mathcal{F}_1$ expressed in $\mathcal{F}_0$
- $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$: axes of $\mathcal{F}_0$.

Rotations

EXAMPLE



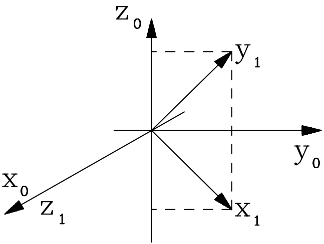
By projecting the unit vectors $\mathbf{i}_1, \mathbf{j}_1, \mathbf{k}_1$ on $\mathbf{i}_0, \mathbf{j}_0, \mathbf{k}_0$, the components of the principal axes of $\mathcal{F}_1$ in $\mathcal{F}_0$ are obtained:

$$\begin{cases} \mathbf{i}_1 = [0, \;\; 1/\sqrt{2}, \;\; -1/\sqrt{2}]^T \\ \mathbf{j}_1 = [0, \;\; 1/\sqrt{2}, \;\; 1/\sqrt{2}]^T \\ \mathbf{k}_1 = [1, \;\; 0, \;\; 0]^T \end{cases}$$

Rotations

The rotation matrix between $\mathcal{F}_0$ and $\mathcal{F}_1$ is obtained from these three vectors:

$$\mathbf{R} = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$



In general

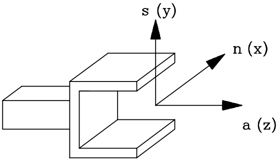
$$\begin{array}{cc} & \begin{array}{ccc} \mathbf{x}_1 & \mathbf{y}_1 & \mathbf{z}_1 \end{array} \\ \begin{array}{c} \mathbf{x}_0 \\ \mathbf{y}_0 \\ \mathbf{z}_0 \end{array} & \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \end{array}$$

Rotations

Usually, in robotics the symbols $\mathbf{n}, \mathbf{s}, \mathbf{a}$, are used to indicate the axes $\mathbf{x}_1, \mathbf{y}_1, \mathbf{z}_1$, then

$$^0\mathbf{R}_1 = [^0\mathbf{n} \quad ^0\mathbf{s} \quad ^0\mathbf{a}] = \begin{bmatrix} n_x & s_x & a_x \\ n_y & s_y & a_y \\ n_z & s_z & a_z \end{bmatrix}$$

defining the relative orientation between $\mathcal{F}_0$ and $\mathcal{F}_1$.



Symbols $\mathbf{n}, \mathbf{s}, \mathbf{a}$ refer to a frame fixed on the end-effector (e.g. gripper) with

- $\mathbf{z}$ axis ($\mathbf{a}$) along the approach direction
- $\mathbf{y}$ axis ($\mathbf{s}$) in the sliding plane of the fingers
- $\mathbf{x}$ axis ($\mathbf{n}$) in the normal direction with respect to $\mathbf{y}, \mathbf{z}$.

Rotations

Rotation matrix:

$$\mathbf{R} = [\mathbf{n} \;\; \mathbf{s} \;\; \mathbf{a}] = \begin{bmatrix} n_x & s_x & a_x \\ n_y & s_y & a_y \\ n_z & s_z & a_z \end{bmatrix}$$

From the conditions:

$$\begin{cases} \mathbf{n}^T \mathbf{a} = \mathbf{s}^T \mathbf{a} = \mathbf{s}^T \mathbf{n} = 0 \\ ||\mathbf{n}|| = ||\mathbf{s}|| = ||\mathbf{a}|| = 1 \end{cases}$$

it follows that $\mathbf{R}$ is an orthonormal matrix, i.e.

$$\mathbf{R} \, \mathbf{R}^T = \mathbf{R}^T \, \mathbf{R} = \mathbf{I}_3 \qquad \mathbf{I}_3\text{: } 3 \times 3 \text{ identity matrix}$$

A rotation matrix is always invertible. By pre-multiplying by $\mathbf{R}^{-1}$ we have

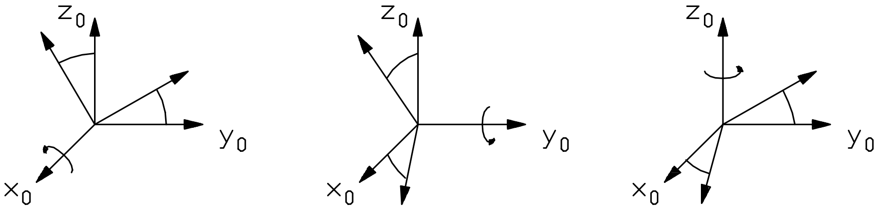
$$\mathbf{R}^{-1} = \mathbf{R}^T$$

i.e.

$$^0\mathbf{R}_1^{-1} = \, ^1\mathbf{R}_0 = \, ^0\mathbf{R}_1^T$$

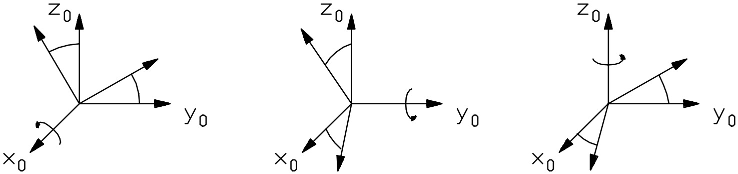
Elementary rotations

Consider two frames $\mathcal{F}_0$ and $\mathcal{F}_1$ with coincident origins.



Rotations of θ about the x_0, y_0, and z_0 axes

Elementary rotations



In the first case, $\mathcal{F}_1$ is obtained with a rotation of an angle θ about the $\mathbf{x}_0$ axis of $\mathcal{F}_0$.

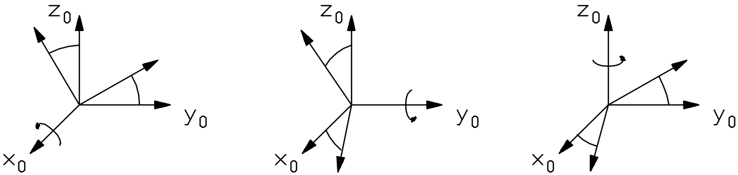
From

$$\mathbf{R} = \begin{bmatrix} {}^0\mathbf{i}_1^T\mathbf{i} & {}^0\mathbf{j}_1^T\mathbf{i} & {}^0\mathbf{k}_1^T\mathbf{i} \\ {}^0\mathbf{i}_1^T\mathbf{j} & {}^0\mathbf{j}_1^T\mathbf{j} & {}^0\mathbf{k}_1^T\mathbf{j} \\ {}^0\mathbf{i}_1^T\mathbf{k} & {}^0\mathbf{j}_1^T\mathbf{k} & {}^0\mathbf{k}_1^T\mathbf{k} \end{bmatrix}$$

we have

$${}^0\mathbf{R}_1 = Rot(\mathbf{x}, \theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

Elementary rotations



Similarly, considering rotations about $\mathbf{y}_0$ and $\mathbf{z}_0$:

$$
{}^0\mathbf{R}_1 = Rot(\mathbf{y}, \theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}
$$

$$
{}^0\mathbf{R}_1 = Rot(\mathbf{z}, \theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

Rotation matrices R relate different reference frames.

Rotations

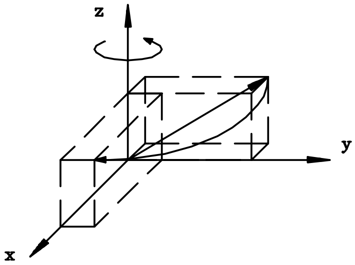
Another interpretation for rotation matrices.

Let us consider a rotation of point ${}^0\mathbf{p}_1 = [7, 3, 2]^T$ by 90^o about $\mathbf{z}_0$.

The matrix expressing the rotation is

$$\mathbf{R}_1 = Rot(\mathbf{z}, 90^o) = \begin{bmatrix} \cos 90^o & -\sin 90^o & 0 \\ \sin 90^o & \cos 90^o & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



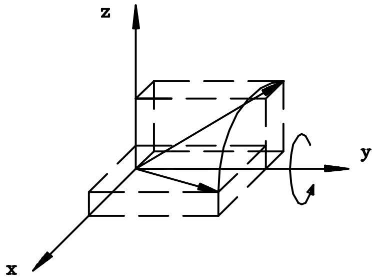
Therefore:

$${}^0\mathbf{p}_2 = \begin{bmatrix} -3 \\ 7 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ 3 \\ 2 \end{bmatrix}$$

Rotations

Consider now a second rotation of 90^o about $\mathbf{y}_0$:

$$^0\mathbf{p}_3 = \begin{bmatrix} 2 \\ 7 \\ 3 \end{bmatrix} = \mathbf{R}_2 \; ^0\mathbf{p}_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -3 \\ 7 \\ 2 \end{bmatrix}$$



Rotations

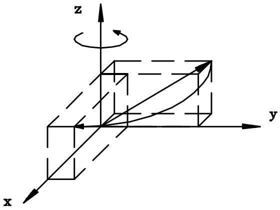
By combining the two rotations one obtains

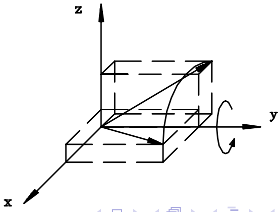
$$\mathbf{R} = \mathbf{R}_2\mathbf{R}_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

from which

$$^{0}\mathbf{p}_3 = \begin{bmatrix} 2 \\ 7 \\ 3 \end{bmatrix} = \mathbf{R}\ ^{0}\mathbf{p}_1 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 7 \\ 3 \\ 2 \end{bmatrix}$$

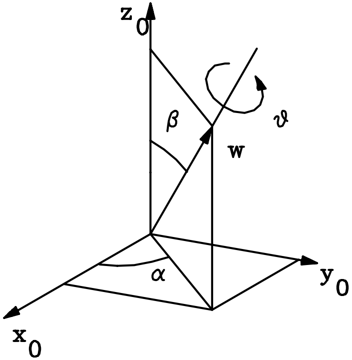
Rotation matrices "rotates" vectors with respect to a fixed reference frame.





Axis/angle rotations

Rotation θ about a generic unit vector $\mathbf{w} = [w_x \ w_y \ w_z]^T$.



The rotation of the angle θ about $\mathbf{w}$ is equivalent to the following procedure:

- Aligne $\mathbf{w}$ with $\mathbf{z}_0$
- Rotate by θ about $\mathbf{w} \equiv \mathbf{z}_0$
- Restore $\mathbf{w}$ in its original position.

Each rotation is performed with respect to $\mathcal{F}_0$, then:

$$Rot(\mathbf{w}, \theta) = Rot(\mathbf{z}_0, \alpha) Rot(\mathbf{y}_0, \beta) Rot(\mathbf{z}_0, \theta) Rot(\mathbf{y}_0, -\beta) Rot(\mathbf{z}_0, -\alpha)$$

Axis/angle rotations

Moreover, since $||\mathbf{w}|| = 1$, we have:

$$\sin\alpha = \frac{w_y}{\sqrt{w_x^2 + w_y^2}} \qquad \cos\alpha = \frac{w_x}{\sqrt{w_x^2 + w_y^2}}$$

$$\sin\beta = \sqrt{w_x^2 + w_y^2} \qquad \cos\beta = w_z$$

The matrix $\mathbf{R}$ representing the rotation is therefore given by

$$\mathbf{R}(\mathbf{w},\theta) = \begin{bmatrix} w_x w_x V_\theta + C_\theta & w_y w_x V_\theta - w_z S_\theta & w_z w_x V_\theta + w_y S_\theta \\ w_x w_y V_\theta + w_z S_\theta & w_y w_y V_\theta + C_\theta & w_z w_y V_\theta - w_x S_\theta \\ w_x w_z V_\theta - w_y S_\theta & w_y w_z V_\theta + w_x S_\theta & w_z w_z V_\theta + C_\theta \end{bmatrix}$$

being $C_\theta = \cos\theta, S_\theta = \sin\theta$, e $V_\theta = \text{vers}\,\theta = 1 - \cos\theta$.

Proprieties of rotations

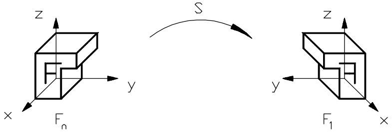
1. Not all the orthogonal matrices ($\mathbf{R}^T\ \mathbf{R} = \mathbf{I}$) for which the following conditions

$$\begin{cases} \mathbf{n}^T\ \mathbf{a}\ =\ \mathbf{s}^T\ \mathbf{a}\ =\ \mathbf{s}^T\ \mathbf{n}\ =\ 0 \\ ||\mathbf{n}|| = ||\mathbf{s}|| = ||\mathbf{a}|| = 1 \end{cases}$$

are satisfied represent rotations. For example, matrix

$$\mathbf{S} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

does not represent a rotation, but rather a "specular" transformation.



It is not possible, starting from $\mathcal{F}_0$, to obtain frame $\mathcal{F}_1$ with a rotation. $\mathcal{F}_1$ may be obtained only by means of a specular reflection.

This is not physically feasible for a rigid body.

Proprieties of rotations

If matrix $\mathbf{R}$ represents a rigid body rotation, then

$$\det(\mathbf{R}) = 1$$

Because of their properties, the rotation matrices in $\mathbb{R}^3$ belong to a "special set", the *Special Orthogonal group of order 3*, i.e. $So(3)$.

More in general, the set of $n \times n$ matrices $\mathbf{R}$ satisfying the two conditions

$$\begin{cases} \mathbf{R}\mathbf{R}^T = \mathbf{R}^T\mathbf{R} = \mathbf{I} \\ \det(\mathbf{R}) = +1 \end{cases}$$

is called $So(n)$: Special Orthogonal group in $\mathbb{R}^n$

$\Longrightarrow$ *Special*: $\det(\mathbf{R}) = +1$

$\Longrightarrow$ *Orthogonal*: $\mathbf{R}\mathbf{R}^T = \mathbf{R}^T\mathbf{R} = \mathbf{I}$

$$So(n) = \{\mathbf{R} \in \mathbb{R}^{n\times n} : \quad \mathbf{R}\mathbf{R}^T = \mathbf{R}^T\mathbf{R} = \mathbf{I}, \quad \det(\mathbf{R}) = +1\}$$

Proprieties of rotations

2. The equations

$$ {}^0\mathbf{R}_1 = [\ {}^0\mathbf{n}\ {}^0\mathbf{s}\ {}^0\mathbf{a}\ ] = \begin{bmatrix} n_x & s_x & a_x \\ n_y & s_y & a_y \\ n_z & s_z & a_z \end{bmatrix} \qquad {}^0\mathbf{R}_1^{-1} = {}^1\mathbf{R}_0 = {}^0\mathbf{R}_1^T $$

allow to consider the relative rotation of two frames, and to transform in $\mathcal{F}_0$ vectors defined in $\mathcal{F}_1$. The expression of ${}^1\mathbf{p}$ in $\mathcal{F}_0$ is given by:

$$ {}^0\mathbf{p} = {}^0\mathbf{R}_1\ {}^1\mathbf{p} = \begin{bmatrix} n_x & s_x & a_x \\ n_y & s_y & a_y \\ n_z & s_z & a_z \end{bmatrix} {}^1\mathbf{p} $$

Proprieties of rotations

The composition of more rotations is expressed by a simple matrix multiplication:

Given $n+1$ reference frames $\mathcal{F}_0$, ..., $\mathcal{F}_n$ with coincident origins and relative orientation expressed by ${}^{i-1}\mathbf{R}_i, \;\; i = 1, \dots n$, and given the vector ${}^n\mathbf{p}$ in $\mathcal{F}_n$, then

$$ {}^0\mathbf{p} = \; {}^0\mathbf{R}_1 \; {}^1\mathbf{R}_2 \dots {}^{n-1}\mathbf{R}_n \; {}^n\mathbf{p} $$

A note about computational complexity:

$$ {}^0\mathbf{p} = \quad ({}^0\mathbf{R}_1 \; {}^1\mathbf{R}_2 \; {}^2\mathbf{R}_3) \; {}^3\mathbf{p} = \; {}^0\mathbf{R}_3 \; {}^3\mathbf{p} \qquad \rightarrow \begin{cases} \text{63 products} \\ \text{42 summations} \end{cases} $$

$$ {}^0\mathbf{p} = \quad \underbrace{({}^0\mathbf{R}_1 \; ({}^1\mathbf{R}_2 \; \underbrace{({}^2\mathbf{R}_3 \; {}^3\mathbf{p})}_{{}^2\mathbf{p}}))}_{{}^1\mathbf{p}} \qquad \rightarrow \begin{cases} \text{27 products} \\ \text{18 summations} \end{cases} $$

3. From $\quad {}^0\mathbf{p} = \; {}^0\mathbf{R}_1 \; {}^1\mathbf{p} \quad$ it follows that a rotation applied to vector ${}^1\mathbf{p}$ is a linear function: $\quad {}^0\mathbf{p} = \mathbf{r}({}^1\mathbf{p}) = \; {}^0\mathbf{R}_1 \; {}^1\mathbf{p}$. $\quad$ Given two vectors $\mathbf{p}, \mathbf{q}$ and two scalar quantities a, b, we have

$$ \mathbf{r}(a\mathbf{p} + b\mathbf{q}) = a\mathbf{r}(\mathbf{p}) + b\mathbf{r}(\mathbf{q}) $$

Proprieties of rotations

4. Rotations do not change the amplitude of a vector:

$$\|\mathbf{Ra}\| = \|\mathbf{a}\|$$

As a matter of fact:

$$\|\mathbf{Ra}\| = \mathbf{a}^T\mathbf{R}^T\mathbf{Ra} = \mathbf{a}^T\mathbf{a} = \|\mathbf{a}\|$$

5. The inner product, and then the angle between two vectors, is invariant with respect to rotations:

$$\mathbf{a}^T\mathbf{b} = (\mathbf{Ra})^T(\mathbf{Rb})$$

As a matter of fact:

$$(\mathbf{Ra})^T(\mathbf{Rb}) = \mathbf{a}^T\mathbf{R}^T\mathbf{Rb} = \mathbf{a}^T\mathbf{b} = \|\mathbf{a}\|\;\|\mathbf{b}\|\cos\theta$$

6. Since $\mathbf{R}$ is an orthogonal matrix, the following property holds

$$\mathbf{R}(\mathbf{a}\times\mathbf{b}) = \mathbf{Ra}\times\mathbf{Rb}$$

Proprieties of rotations

7. In general, the product of rotation matrices does not commute:

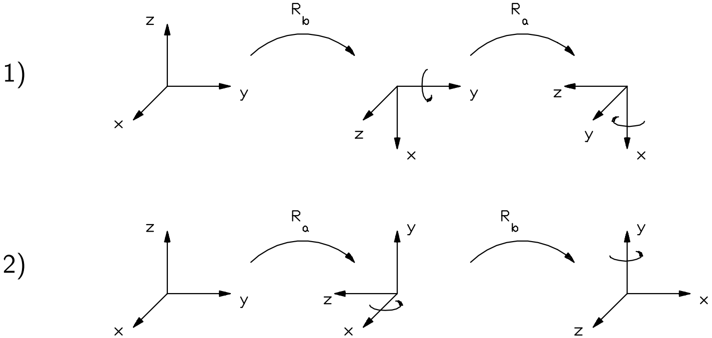
$$\mathbf{R}_a\mathbf{R}_b \neq \mathbf{R}_b\mathbf{R}_a$$

Except the trivial case of the identity matrix (i.e. when $\mathbf{R} = \mathbf{I}_3$), rotations commute only if the rotation axis is the same!

Consider the two rotations by a 90^o angle about the x_0 and y_0 axes:

$$\mathbf{R}_a = Rot(\mathbf{x}, 90^0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \mathbf{R}_b = Rot(\mathbf{y}, 90^o) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

Proprieties of rotations



Case 1) $\mathbf{R}_b$ followed by $\mathbf{R}_a$:

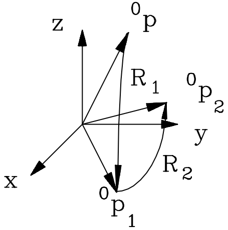
$$\mathbf{R} = \mathbf{R}_b \mathbf{R}_a = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}$$

Case 2) $\mathbf{R}_a$ followed by $\mathbf{R}_b$:

$$\mathbf{R} = \mathbf{R}_a \mathbf{R}_b = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Proprieties of rotations

8. It may be of interest to define a sequence of rotations with respect to $\mathcal{F}_0$, and not with respect to the current frame $\mathcal{F}_i$ as assumed until now.



Consider two rotations $\mathbf{R}_1 = Rot(\mathbf{y}, \phi)$ and $\mathbf{R}_2 = Rot(\mathbf{z}, \theta)$ about the axes $\mathbf{y}_0$ and $\mathbf{z}_0$ of $\mathcal{F}_0$.

What is the result of applying first $\mathbf{R}_1$ and then $\mathbf{R}_2$?

Consider the vector ${}^0\mathbf{p}$ in $\mathcal{F}_0$. After the first rotation $\mathbf{R}_1$, the new expression of the vector (still wrt $\mathcal{F}_0$) is

$$^0\mathbf{p}_1 = \mathbf{R}_1 \, {}^0\mathbf{p}$$

Since also the second rotation is about an axis of $\mathcal{F}_0$, we have

$$^0\mathbf{p}_2 = \mathbf{R}_2 \, {}^0\mathbf{p}_1 = \mathbf{R}_2 \, \mathbf{R}_1 \, {}^0\mathbf{p}$$

More in general, given n consecutive rotations $\mathbf{R}_i, i = 1, \ldots, n$ defined with respect to the same reference frame $\mathcal{F}_0$, then

$$^0\mathbf{p}_n = \mathbf{R}_n \, \mathbf{R}_{n-1} \ldots \mathbf{R}_1 \, {}^0\mathbf{p}$$

Proprieties of rotations

Then, there are two different possibilities to define a sequence of consecutive rotations:

1. If each rotation is expressed wrt the current frame $\mathcal{F}_n$, $\mathcal{F}_{n-1}, \ldots, \mathcal{F}_0$, then the equivalent rotation matrix ${}^0\mathbf{R}_n$ is obtained by post-multiplication of the matrices ${}^{i-1}\mathbf{R}_i$.

$$ {}^0\mathbf{p} = {}^0\mathbf{R}_1 \; {}^1\mathbf{R}_2 \; \ldots \; {}^{n-1}\mathbf{R}_n \; {}^n\mathbf{p} $$

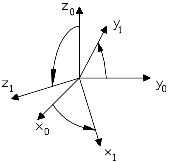
2. If matrices $\mathbf{R}_i, i = 1, \ldots, n$ describe rotations about an axis of the base frame $\mathcal{F}_0$, the equivalent matrix is obtained by pre-multiplication of the matrices.

$$ {}^0\mathbf{p}_n = \mathbf{R}_n \; \mathbf{R}_{n-1} \; \ldots \; \mathbf{R}_1 \; {}^0\mathbf{p} $$

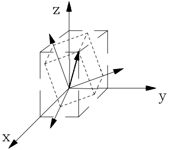
Interpretations of a rotation matrix

In summary, a rotation matrix ${}^0\mathbf{R}_1$ has three equivalent interpretations:

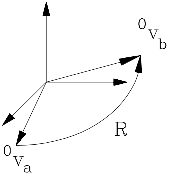
1. ${}^0\mathbf{R}_1$ describes the mutual orientation of two reference frames $\mathcal{F}_0$ and $\mathcal{F}_1$; the columns of ${}^0\mathbf{R}_1$ are the direction cosines of the axes of $\mathcal{F}_1$ expressed in $\mathcal{F}_0$



2. ${}^0\mathbf{R}_1$ defines the coordinate transformation between the coordinates of a point expressed in $\mathcal{F}_0$ and in $\mathcal{F}_1$ (common origin)



3. ${}^0\mathbf{R}_1$ rotates a vector ${}^0\mathbf{v}_a$ to ${}^0\mathbf{v}_b$ in a given reference frame $\mathcal{F}_0$



Representations of rotations

A rotation is described with a 3×3 matrix with 9 elements:

$$\mathbf{R} = \begin{bmatrix} n_x & s_x & a_x \\ n_y & s_y & a_y \\ n_z & s_z & a_z \end{bmatrix}$$

On the other hand, a rigid body in $\mathbb{R}^3$ has 3 rotational dof $\rightarrow$ Three parameters should be sufficient to describe its orientation

A 3×3 matrix, although computationally efficient, is **redundant**. Among the 9 elements of $\mathbf{R}$ one can define the following relations:

$$\begin{cases} \mathbf{n}^T \mathbf{a} = \mathbf{s}^T \mathbf{a} = \mathbf{s}^T \mathbf{n} = 0 \\ ||\mathbf{n}|| = ||\mathbf{s}|| = ||\mathbf{a}|| = 1 \end{cases}$$

Note that it is sufficient to know 6 elements of $\mathbf{R}$ to define completely the matrix. If only 5 (or less) elements are known, $\mathbf{R}$ cannot be determined univocally.

Representations of rotations

Theoretically, only 3 parameters are sufficient to describe the orientation of a rigid body in the 3D space.

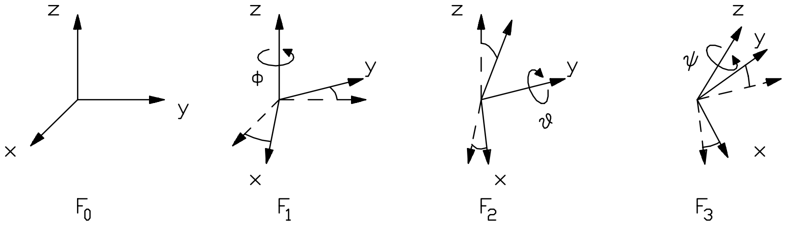
There are representations based on 3 parameters only *(minimal representations)*, more “compact” than rotation matrices, although computationally less convenient.

Among these representations, we have:

- Euler angles: three consecutive rotations about axes $\mathbf{z}, \mathbf{y}', \mathbf{z}''$
- Roll, Pitch and Yaw angles: three consecutive rotations about axes $\mathbf{z}_0, \mathbf{y}_0, \mathbf{x}_0$
- Axis/Angle representation: a unitary rotation axis $\mathbf{r}$ and the angle θ

Euler angles

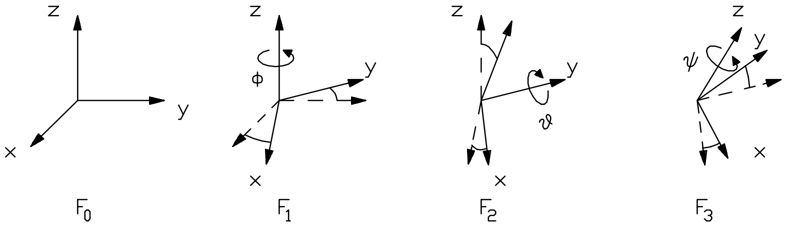
Euler angles (ϕ, θ, ψ) represents three rotations, applied sequentially about the axes $\mathbf{z}_0, \mathbf{y}_1, \mathbf{z}_2$ of the *current* frame $\mathcal{F}_0$, $\mathcal{F}_1$, $\mathcal{F}_2$.



Consider a base frame $\mathcal{F}_0$. By applying the three rotations we have

- A frame $\mathcal{F}_1$ obtained with the rotation ϕ about $\mathbf{z}_0$
- A frame $\mathcal{F}_2$ obtained from $\mathcal{F}_1$ with the rotation θ about $\mathbf{y}_1$
- A frame $\mathcal{F}_3$ obtaine from $\mathcal{F}_2$ with the rotation ψ about $\mathbf{z}_2$

Euler angles



By composing the three rotations, the total rotation from $\mathcal{F}_0$ to $\mathcal{F}_3$ is

$$
\begin{aligned}
{}^0\mathbf{R}_3 = \mathbf{R}_{Euler}(\phi,\theta,\psi) &= Rot(\mathbf{z}_0,\phi)Rot(\mathbf{y}_1,\theta)Rot(\mathbf{z}_2,\psi) \\
&= \begin{bmatrix} C_\phi C_\theta C_\psi - S_\phi S_\psi & -C_\phi C_\theta S_\psi - S_\phi C_\psi & C_\phi S_\theta \\ S_\phi C_\theta C_\psi + C_\phi S_\psi & -S_\phi C_\theta S_\psi + C_\phi C_\psi & S_\phi S_\theta \\ -S_\theta C_\psi & S_\theta S_\psi & C_\theta \end{bmatrix}
\end{aligned}
$$

Euler angles

Rotation matrix corresponding to the Euler angles:

$$\mathbf{R}_{Euler}(\phi,\theta,\psi) = \left[\begin{array}{ccc} C_\phi C_\theta C_\psi - S_\phi S_\psi & -C_\phi C_\theta S_\psi - S_\phi C_\psi & C_\phi S_\theta \\ S_\phi C_\theta C_\psi + C_\phi S_\psi & -S_\phi C_\theta S_\psi + C_\phi C_\psi & S_\phi S_\theta \\ -S_\theta C_\psi & S_\theta S_\psi & C_\theta \end{array}\right]$$

Inverse problem: compute the Euler angles corresponding to a given rotation matrix $\mathbf{R}$:

$$\mathbf{R} = \left[\begin{array}{ccc} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{array}\right] \quad \rightarrow \quad (\phi,\ \theta,\ \psi) \quad ?$$

Atan2 function

- $\arctan(\frac{y}{x}) = \arctan(\frac{-y}{-x})$, gives results in two quadrants (in $[-\pi/2, \ +\pi/2]$)
- atan2 is the arctangent with output values in the four quadrants:
  - two input arguments
  - gives values in $[-\pi, \ +\pi]$
  - undefined only for (0, 0)
- uses the sign of both arguments to define the output quadrant
- based on arctan function with output values in $[-\pi/2, \ +\pi/2]$
- available in main languages (C++, Matlab, ...)

$$\text{atan2}\,(y, x) \quad = \quad \begin{cases} \arctan(y/x) & x > 0 \\ \pi + \arctan(y/x) & y \geq 0, x < 0 \\ -\pi + \arctan(y/x) & y < 0, x < 0 \\ \pi/2 & y > 0, x = 0 \\ -\pi/2 & y < 0, x = 0 \\ \text{undefined} & y = 0, x = 0 \end{cases}$$

Euler angles

Two cases are possible:

1. $r_{13}^2 + r_{23}^2 \neq 0 \quad \rightarrow \quad \sin\theta \neq 0$. By assuming $0 < \theta < \pi$ ($\sin\theta > 0$), one obtains:

$$\begin{cases} \phi = \text{atan2}\,(r_{23}, r_{13}); \\ \theta = \text{atan2}\,(\sqrt{r_{13}^2 + r_{23}^2}, r_{33}); \\ \psi = \text{atan2}\,(r_{32}, -r_{31}) \end{cases}$$

or, with $-\pi < \theta < 0$, ($\sin\theta < 0$):

$$\begin{cases} \phi = \text{atan2}\,(-r_{23}, -r_{13}); \\ \theta = \text{atan2}\,(-\sqrt{r_{13}^2 + r_{23}^2}, r_{33}); \\ \psi = \text{atan2}\,(-r_{32}, r_{31}) \end{cases}$$

Two possible sets of solutions depending on the sign of $\sin\theta$.

Euler angles

2. $r_{13}^2 + r_{23}^2 = 0$ $\quad(\theta = 0, \pi$ and $\cos\theta = \pm 1)$. By choosing $\theta = 0$ $(\cos\theta = 1)$ one obtains

$$\begin{cases} \theta = 0 \\ \phi + \psi = \text{atan2}\,(r_{21}, r_{11}) = \text{atan2}\,(-r_{12}, r_{11}); \end{cases}$$

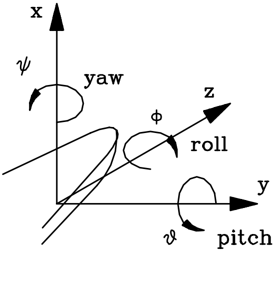
On the other hand, if $\theta = \pi$ $(\cos\theta = -1)$

$$\begin{cases} \theta = \pi \\ \phi - \psi = \text{atan2}\,(-r_{21}, -r_{11}) = \text{atan2}\,(-r_{12}, -r_{11}); \end{cases}$$

In both cases, infinite solutions are obtained (only the sum or difference of ϕ and θ is known).

Being $\theta = 0, \pi$, the rotations by the angles ϕ and ψ occur about parallel (the same) axes, i.e. the **z** axis.

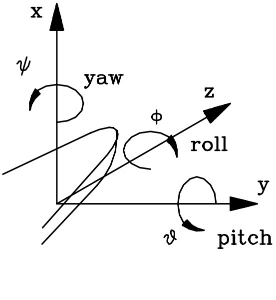
Roll, Pitch, Yaw



Consider three consecutive rotations about the axes of the base frame $\mathcal{F}_0$:

- A rotation ψ about $\mathbf{x}_0$, (*yaw*),
- A rotation θ about $\mathbf{y}_0$, (*pitch*)
- A rotation ϕ about $\mathbf{z}_0$, (*roll*).

Roll, Pitch, Yaw



By properly composing the three rotations:

$$
\begin{aligned}
{}^0\mathbf{R}_3 = \mathbf{R}_{RPY}(\phi,\theta,\psi) &= Rot(\mathbf{z}_0,\phi)Rot(\mathbf{y}_0,\theta)Rot(\mathbf{x}_0,\psi) \\
&= \begin{bmatrix} C_\phi C_\theta & -S_\phi C_\psi + C_\phi S_\theta S_\psi & S_\phi S_\psi + C_\phi S_\theta C_\psi \\ S_\phi C_\theta & C_\phi C_\psi + S_\phi S_\theta S_\psi & -C_\phi S_\psi + S_\phi S_\theta C_\psi \\ -S_\theta & C_\theta S_\psi & C_\theta C_\psi \end{bmatrix}
\end{aligned}
$$

Roll, Pitch, Yaw

Rotation matrix corresponding to the RPY angles:

$$\mathbf{R}_{RPY}(\phi,\theta,\psi) = \begin{bmatrix} C_\phi C_\theta & -S_\phi C_\psi + C_\phi S_\theta S_\psi & S_\phi S_\psi + C_\phi S_\theta C_\psi \\ S_\phi C_\theta & C_\phi C_\psi + S_\phi S_\theta S_\psi & -C_\phi S_\psi + S_\phi S_\theta C_\psi \\ -S_\theta & C_\theta S_\psi & C_\theta C_\psi \end{bmatrix}$$

Inverse problem: compute the RPY angles corresponding to a given rotation matrix $\mathbf{R}$:

$$\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \quad \rightarrow \quad (\phi,\ \theta,\ \psi)\ \ (?)$$

Roll, Pitch, Yaw

Two cases are possible:

1. $r_{11}^2 + r_{21}^2 \neq 0 \;\rightarrow\; \cos\theta \neq 0$. By choosing $\theta \in [-\pi/2, \pi/2]$, one obtains:

$$\begin{cases} \phi = \text{atan2}\,(r_{21}, r_{11}); \\ \theta = \text{atan2}\,(-r_{31}, \sqrt{r_{32}^2 + r_{33}^2}); \\ \psi = \text{atan2}\,(r_{32}, r_{33}); \end{cases}$$

Otherwise, if $\theta \in [\pi/2, 3\pi/2]$:

$$\begin{cases} \phi = \text{atan2}\,(-r_{21}, -r_{11}); \\ \theta = \text{atan2}\,(-r_{31}, -\sqrt{r_{32}^2 + r_{33}^2}); \\ \psi = \text{atan2}\,(-r_{32}, -r_{33}); \end{cases}$$

Roll, Pitch, Yaw

2. $r_{11}^2 + r_{21}^2 = 0 \;\rightarrow\; \cos\theta = 0$: $\theta = \pm\pi/2$ and infinite solutions are possible (sum or difference of ψ and ϕ).

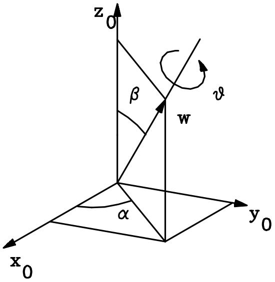
   It may be convenient to (arbitrarily) assign a value (e.g. $\pm 90^o$) to one of the two angles (ϕ or ψ) a $\pm 90^o$ and then compute the remaining one:

$$\begin{cases} \theta = \pm\pi/2; \\ \phi - \psi = \text{atan2}\,(r_{23}, r_{13}) = \text{atan2}\,(-r_{12}, r_{22}); \end{cases}$$

Angle/Axis representation

It is possible to describe any rotation in 3D by means of the rotation angle θ and the corresponding rotation axis $\mathbf{w}$

$$\mathbf{R} = \begin{bmatrix} w_x^2(1-C_\theta)+C_\theta & w_x w_y(1-C_\theta)-w_z S_\theta & w_x w_z(1-C_\theta)+w_y S_\theta \\ w_x w_y(1-C_\theta)+w_z S_\theta & w_y^2(1-C_\theta)+C_\theta & w_y w_z(1-C_\theta)-w_x S_\theta \\ w_x w_z(1-C_\theta)-w_y S_\theta & w_y w_z(1-C_\theta)+w_x S_\theta & w_z^2(1-C_\theta)+C_\theta \end{bmatrix}$$



4 parameters: w_x, w_y, w_z, θ

with the condition: $w_x^2 + w_y^2 + w_z^2 = 1$

$\Longrightarrow$ 3 dof

Angle/Axis representation

Inverse problem: compute the axis **w** and the angle θ corresponding to a given rotation matrix **R**:

$$\theta = \operatorname{acos} \frac{r_{11} + r_{22} + r_{33} - 1}{2}$$

$$\mathbf{w} = \frac{1}{2\sin\theta} \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

The trace of a rotation matrix depends only on the (cosine of) rotation angle.

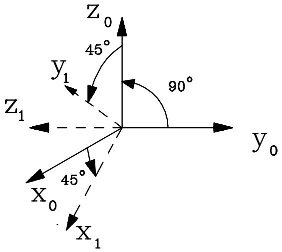
This representation suffers of some drawbacks:

- It is not unique: $Rot(\mathbf{w}, \theta) = Rot(-\mathbf{w}, -\theta)$ (we can arbitrarily assume $0 \leq \theta \leq \pi$)
- If $\theta = 0$ then $Rot(\mathbf{w}, 0) = \mathbf{I}_3$ and **w** is indefinite
- There are numerical problems if $\theta \approx 0$: in this case $\sin\theta \approx 0$ and problems may arise in computing **w**

Example

Compute the rotation matrix corresponding to the RPY angles $0^o, 45^o, 90^o$, i.e.:

1. A rotation of 90^o about x_0, (yaw)
2. A rotation of 45^oabout y_0, (pitch)
3. A rotation of 0^o about z_0, (roll)



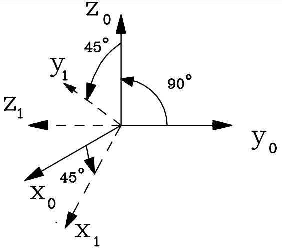
RPY rotations: roll $= 0^o$, pitch $= 45^o$, yaw $= 90^o$

One obtains:

$$\mathbf{R} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

Example

$$\mathbf{R} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$



RPY rotations: roll = 0^o, pitch = 45^o, yaw = 90^o

- The Euler angles corresponding to this rotation matrix are

$$\phi = -90^o; \quad \theta = 90^o; \quad \psi = 45^o$$

- Considering the Angle/Axis representation, one obtains:

rotation of the angle $\theta = 98.42^o$ about the axis $\mathbf{w} = [0.863,\ 0.357,\ -0.357]^T$

Ridig Body Motion – Homogeneous Transformations

Translations

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Translations

Rotations between two coordinate frames can be expressed in matrix form:

$$^{0}\mathbf{p} = \mathbf{R}\ ^{1}\mathbf{p}$$

This is not possible for translations! $\rightarrow$ It is not possible to define a 3×3 matrix $\mathbf{P}$ so that a translation can be expressed as a matrix/vector multiplication

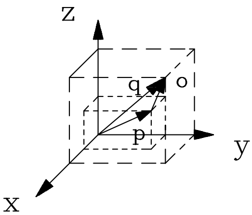
$$^{0}\mathbf{p} = \mathbf{P}\ ^{1}\mathbf{p} \quad \Longrightarrow \quad \textit{not possible!!}$$

A translation of a vector $^{0}\mathbf{p}$ by $^{0}\mathbf{o}$ corresponds to a vectorial summation

$$^{0}\mathbf{q} = {}^{0}\mathbf{p} + {}^{0}\mathbf{o}$$

Then

$$^{0}q_x = {}^{0}p_x + {}^{0}o_x \qquad {}^{0}q_y = {}^{0}p_y + {}^{0}o_y \qquad {}^{0}q_z = {}^{0}p_z + {}^{0}o_z$$



Translations

Then, a translation is expressed as a function $\mathbf{t}(\mathbf{p}) = \mathbf{p} + \mathbf{o}$

In general: $\mathbf{t}(a\mathbf{p} + b\mathbf{q}) = a\mathbf{p} + b\mathbf{q} + \mathbf{o} \neq a\mathbf{t}(\mathbf{p}) + b\mathbf{t}(\mathbf{q})$

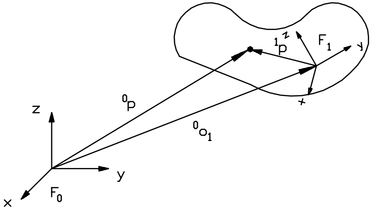
Then, translations are not linear transformations!

The most general transformation between two coordinate frames cannot be represented by a 3×3 matrix.

The composition of a rotation and a translation is obtained from

$$^0\mathbf{q} = {}^0\mathbf{p} + {}^0\mathbf{o}$$

by considering vector $\mathbf{p}$ defined in $\mathcal{F}_1$, rotated and translated with respect to $\mathcal{F}_0$.

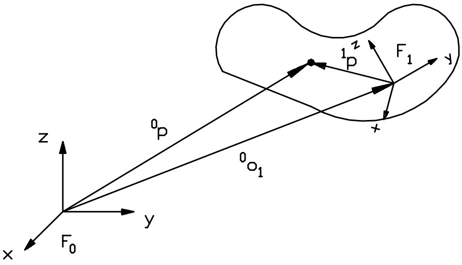


Translations

Since vectors can be added only if they are defined with respect to the same coordinate system:

$$ {}^{0}\mathbf{p} = {}^{0}\mathbf{R}_1 \, {}^{1}\mathbf{p} + {}^{0}\mathbf{o}_1 $$

being ${}^{0}\mathbf{o}_1$ the translation from $\mathcal{F}_0$ to $\mathcal{F}_1$ and ${}^{0}\mathbf{R}_1$ the mutual rotation.



Ridig Body Motion – Homogeneous Transformations

Homogeneous transformations

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Homogeneous transformations

It is of interest to put in matrix form the equation

$$^{0}\mathbf{p} = {}^{0}\mathbf{R}_1 \, {}^{1}\mathbf{p} + {}^{0}\mathbf{o}_1$$

since, in case of successive transformations, one could obtain expressions similar to

$$^{0}\mathbf{p} = {}^{0}\mathbf{R}_1 \, {}^{1}\mathbf{R}_2 \ldots {}^{n-1}\mathbf{R}_n \, {}^{n}\mathbf{p}$$

For this purpose, it is possible to add to matrix $\mathbf{R}$ the vector $^{0}\mathbf{o}$ as fourth column; in this manner a 3×4 matrix is obtained

$$\mathbf{M} = [\mathbf{n} \; \mathbf{s} \; \mathbf{a} \; {}^{0}\mathbf{o}_1] = [\mathbf{R} \; {}^{0}\mathbf{o}_1]$$

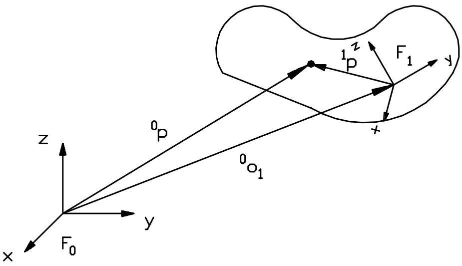
A square matrix is obtained by adding to $\mathbf{M}$ the row $[0 \; 0 \; 0 \; 1]$.

Homogeneous transformations

The *Homogeneous Transformation Matrix* ${}^0\mathbf{T}_1$ is obtained

$$
{}^0\mathbf{T}_1 = \left[\begin{array}{ccc|c} & {}^0\mathbf{R}_1 & & {}^0\mathbf{o}_1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} n_x & s_x & a_x & o_x \\ n_y & s_y & a_y & o_y \\ n_z & s_z & a_z & o_z \\ \hline 0 & 0 & 0 & 1 \end{array}\right]
$$

This matrix represents the transformation (rotation-translation) between $\mathcal{F}_0$ and $\mathcal{F}_1$.



Homogeneous transformations

By defining the *homogeneous* 4-dimensional (in $\mathbb{R}^4$) vector:

$$[\mathbf{p}^T \;\; 1]^T = [p_x \;\; p_y \;\; p_z \;\; 1]^T$$

one obtains

$$\begin{bmatrix} {}^0\mathbf{p} \\ 1 \end{bmatrix} = {}^0\mathbf{T}_1 \begin{bmatrix} {}^1\mathbf{p} \\ 1 \end{bmatrix} = \begin{bmatrix} {}^0\mathbf{R}_1 \, {}^1\mathbf{p} + {}^0\mathbf{o}_1 \\ 1 \end{bmatrix}$$

The subspace $\mathbb{R}^3$ defined by the first three components represents the transformation ${}^0\mathbf{p} = {}^0\mathbf{R}_1 \, {}^1\mathbf{p} + {}^0\mathbf{o}_1$ on vectors in $\mathbb{R}^3$, and physically represents the rigid body motion from $\mathcal{F}_1$ to $\mathcal{F}_0$; the fourth component is not affected by the matrix multiplication.

Homogeneous transformations

Given the homogeneous transformation matrices ${}^0\mathbf{T}_1$, from $\mathcal{F}_1$ to $\mathcal{F}_0$, and ${}^1\mathbf{T}_2$, from $\mathcal{F}_2$ to $\mathcal{F}_1$, the composition

$$ {}^0\mathbf{T}_2 = {}^0\mathbf{T}_1 \; {}^1\mathbf{T}_2 $$

can be applied to vectors $[\ {}^2\mathbf{p}^T \ 1]^T$ defined in $\mathcal{F}_2$, and the result is

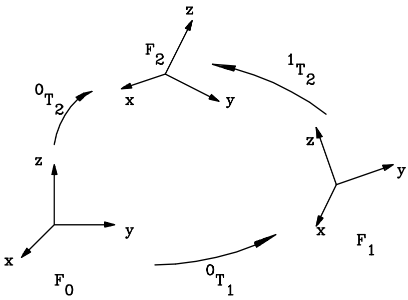
$$ \begin{aligned} {}^0\mathbf{T}_1({}^1\mathbf{T}_2 \left[\begin{array}{c} {}^2\mathbf{p} \\ 1 \end{array} \right]) &= {}^0\mathbf{T}_1 \left[\begin{array}{c} {}^1\mathbf{R}_2 \, {}^2\mathbf{p} + {}^1\mathbf{o}_2 \\ 1 \end{array} \right] \\ &= \left[\begin{array}{c} {}^0\mathbf{R}_1 \, {}^1\mathbf{R}_2 \, {}^2\mathbf{p} + {}^0\mathbf{R}_1 \, {}^1\mathbf{o}_2 + {}^0\mathbf{o} \\ 1 \end{array} \right] \end{aligned} $$

Homogeneous transformations

This is equivalent to the product of the homogeneous matrix

$$ {}^0\mathbf{T}_2 = \begin{bmatrix} & {}^0\mathbf{R}_1\ {}^1\mathbf{R}_2 & & {}^0\mathbf{R}_1\ {}^1\mathbf{o}_2 + {}^0\mathbf{o} \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^0\mathbf{T}_1\ {}^1\mathbf{T}_2 $$

with the vector $[\,{}^2\mathbf{p}^T\ 1]^T$.



In general we have

$$ {}^0\mathbf{T}_n = {}^0\mathbf{T}_1\ {}^1\mathbf{T}_2\ \ldots\ {}^{n-1}\mathbf{T}_n $$

Homogeneous transformation of coordinates

The coordinate transformation bewteen two reference frames $\mathcal{F}_0$ and $\mathcal{F}_1$ may be expressed by a 4×4 matrix ${}^0\mathbf{T}_1$:

$$ {}^0\mathbf{p} = {}^0\mathbf{T}_1 \, {}^1\mathbf{p} $$

Of particular interest are the elementary transformations, i.e. simple rotations or translations along the coordinate axes.
All the coordinate transformations may be obtained by combinations of these elementary transformations.

Elementary rotations and translations

The homogeneous transformation matrices corresponding to rotations of an angle θ about the axes $\mathbf{x}, \mathbf{y}, \mathbf{z}$ of $\mathcal{F}_0$ are:

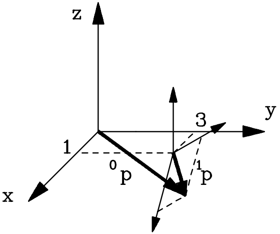
$$Rot(\mathbf{x},\theta)=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C_\theta & -S_\theta & 0 \\ 0 & S_\theta & C_\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, Rot(\mathbf{y},\theta)=\begin{bmatrix} C_\theta & 0 & S_\theta & 0 \\ 0 & 1 & 0 & 0 \\ -S_\theta & 0 & C_\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, Rot(\mathbf{z},\theta)=\begin{bmatrix} C_\theta & -S_\theta & 0 & 0 \\ S_\theta & C_\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

being $C_\theta = \cos\theta, S_\theta = \sin\theta$.

The homogeneous transformation $\mathbf{T}$ corresponding to the translation of vector $\mathbf{v} = [v_x \;\; v_y \;\; v_z]^T$ is

$$\mathbf{T} = Trasl(\mathbf{v}) = \begin{bmatrix} 1 & 0 & 0 & v_x \\ 0 & 1 & 0 & v_y \\ 0 & 0 & 1 & v_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example



The frame $\mathcal{F}_1$, with respect to $\mathcal{F}_0$, is translated of 1 along $\mathbf{x}_0$ and of 3 along $\mathbf{y}_0$, moreover, it is rotated by 30^o about $\mathbf{z}_0$.

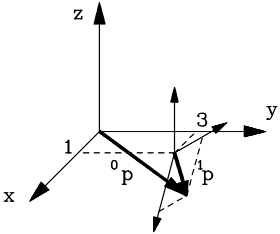
The transformation matrix from $\mathcal{F}_1$ to $\mathcal{F}_0$ is

$$ {}^0\mathbf{T}_1 = \begin{bmatrix} 0.866 & -0.500 & 0 & 1 \\ 0.500 & 0.866 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Example

Consider a point defined in $\mathcal{F}_1$ by

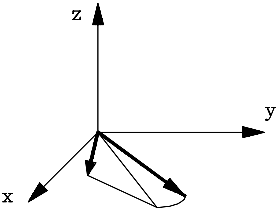
$$^1\mathbf{p} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$



Its coordinates in $\mathcal{F}_0$ are

$$^0\mathbf{p} = \, ^0\mathbf{T}_1 \, ^1\mathbf{p} = \begin{bmatrix} 0.866 & -0.500 & 0 & 1 \\ 0.500 & 0.866 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.232 \\ 4.866 \\ 0 \\ 1 \end{bmatrix}$$

Example



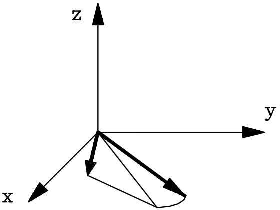
Consider the point in $\mathcal{F}_0$

$$^0\mathbf{p}_a = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Let us apply to $^0\mathbf{p}$ a translation of 1 along $\mathbf{x}_0$, of 3 along $\mathbf{y}_0$; then rotate the vector by 30^o about $\mathbf{z}_0$. The result is obtained by multiplying vector $^0\mathbf{p}$ by the matrix

$$\mathbf{T} = \begin{bmatrix} 0.866 & -0.500 & 0 & 1 \\ 0.500 & 0.866 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example



Then

$$
{}^0\mathbf{p}_b = \mathbf{T}\ {}^0\mathbf{p}_a = \mathbf{T} = \begin{bmatrix} 0.866 & -0.500 & 0 & 1 \\ 0.500 & 0.866 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.232 \\ 4.866 \\ 0 \\ 1 \end{bmatrix}
$$

The same numerical result is obtained, although the physical interpretation is different.

Interpretations of a homogeneous transformation matrix

Similarly to rotation matrices, also an homogeneous transformation matrix ${}^0\mathbf{T}_1$

$$ {}^0\mathbf{T}_1 = \begin{bmatrix} {}^0\mathbf{R}_1 & {}^0\mathbf{v} \\ 0\ 0\ 0 & 1 \end{bmatrix} $$

has three possible physical interpretations:

1. Description of $\mathcal{F}_1$ in $\mathcal{F}_0$: in particular ${}^0\mathbf{v}$ represents the origin of $\mathcal{F}_1$ with respect to $\mathcal{F}_0$, and the elements of ${}^0\mathbf{R}_1$ give the direction of the axes of $\mathcal{F}_1$
2. Coordinate transformation of vectors between $\mathcal{F}_1$ and $\mathcal{F}_0$, ${}^1\mathbf{p} \rightarrow {}^0\mathbf{p}$;
3. Translates and rotates a generic vector ${}^0\mathbf{p}_a$ to ${}^0\mathbf{p}_b$ in a given reference frame $\mathcal{F}_0$

Inverse transformation

Once the position/orientation of $\mathcal{F}_1$ with respect to $\mathcal{F}_0$ are known, defined by the homogeneous transformation matrix ${}^0\mathbf{T}_1$, it is simple to compute the inverse transformation ${}^1\mathbf{T}_0 = ({}^0\mathbf{T}_1)^{-1}$, defining the position/orientation of $\mathcal{F}_0$ with respect to $\mathcal{F}_1$.

From

$$ {}^0\mathbf{T}_1 \, {}^1\mathbf{T}_0 = \mathbf{I}_4 $$

it follows that

$$ {}^1\mathbf{T}_0 = \left[\begin{array}{ccc|c} & {}^0\mathbf{R}_1^T & & -{}^0\mathbf{R}_1^T \mathbf{v} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} n_x & n_y & n_z & -\mathbf{v}^T\mathbf{n} \\ s_x & s_y & s_z & -\mathbf{v}^T\mathbf{s} \\ a_x & a_y & a_z & -\mathbf{v}^T\mathbf{a} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] $$

Inverse transformation

As a matter of fact, if

$$ {}^{1}\mathbf{T}_0 = \begin{bmatrix} & \mathbf{M} & & \mathbf{x} \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

then

$$ \begin{bmatrix} & \mathbf{R} & & \mathbf{v} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} & \mathbf{M} & & \mathbf{x} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

and therefore

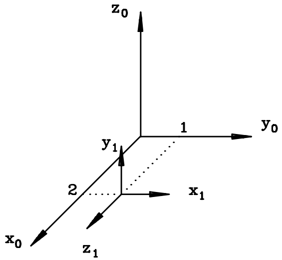
$$ \begin{array}{rcl} \mathbf{I}_3 & = & \mathbf{R}\mathbf{M} \\ \mathbf{0} & = & \mathbf{R}\mathbf{x} + \mathbf{v} \end{array} \quad \Longrightarrow \quad \begin{array}{rcl} \mathbf{M} & = & \mathbf{R}^T \\ \mathbf{x} & = & -\mathbf{R}^T\mathbf{v} \end{array} $$

Example

Given

$$\mathbf{T} = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

compute its inverse transformation.



Solution:

$$\mathbf{T}^{-1} = \left[\begin{array}{ccc|c} & \mathbf{R}^T & & -\mathbf{R}^T\mathbf{v} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} n_x & n_y & n_z & -\mathbf{v}^T\mathbf{n} \\ s_x & s_y & s_z & -\mathbf{v}^T\mathbf{s} \\ a_x & a_y & a_z & -\mathbf{v}^T\mathbf{a} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Homogeneous transformations

- The term *homogeneous* derives from projective geometry.
- Equations describing lines and planes in projective geometry in $\mathbb{R}^3$ are homogeneous in the four variables x_1, x_2, x_3, x_4 in $\mathbb{R}^4$.
- In $\mathbb{R}^3$ these equations, *affine transformations*, are non homogeneous in x_1, x_2, x_3, (lines or planes not passing through the origin present a constant term - non function of x_1, x_2, x_3 - in their expression).
- In general, computations of an affine transformations in $\mathbb{R}^{n-1}$ may be expressed as homogeneous linear transformations in $\mathbb{R}^n$:

$$
\begin{aligned}
{}^0\mathbf{p} &= {}^0\mathbf{R}_1\, {}^1\mathbf{p} + {}^0\mathbf{o}_1 && \text{affine transformation} \\
{}^0\mathbf{p}' &= \left[\begin{array}{c} {}^0\mathbf{p} \\ \hline 1 \end{array}\right] = \left[\begin{array}{c|c} {}^0\mathbf{R}_1 & {}^0\mathbf{o}_1 \\ \hline \mathbf{0} & 1 \end{array}\right] \left[\begin{array}{c} {}^1\mathbf{p} \\ \hline 1 \end{array}\right] && \text{homogeneous transformation}
\end{aligned}
$$

Homogeneous transformations

The most general expression for an homogeneous transformation is

$$\mathbf{T} = \left[\begin{array}{c|c} \mathbf{D}_{3\times 3} & \mathbf{p}_{3\times 1} \\ \hline \mathbf{f}_{1\times 3} & s \end{array}\right] = \left[\begin{array}{c|c} \text{Deformation} & \text{Translation} \\ \hline \text{Perspective} & \text{Scale} \end{array}\right]$$

Note the terms $\mathbf{D}$, $\mathbf{f}_{1\times 3}$ and s.

These quantities, in robotics, are alway assumed as:

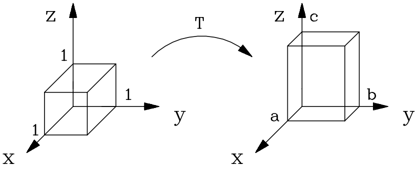
- A rotation matrix ($\mathbf{D} = \mathbf{R}$),
- The null vector $[0 \;\; 0 \;\; 0]$ ($\mathbf{f}_{1\times 3} = [0 \;\; 0 \;\; 0]$),
- The unit gain ($s = 1$)

In other cases (e.g. computer graphics), these quantities are used to obtain deformations, perspective distortions, change of scaling factors, and so on (in general: effects non applicable to rigid bodies).

Example

Consider the transformation matrix

$$\mathbf{T} = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



This transformation applies a different "gain" along the three reference axes $\mathbf{x}, \mathbf{y}, \mathbf{z}$. A deformation of the body is obtained.

Point $\mathbf{p} = [1, 1, 1]^T$ is transformed to

$$\mathbf{p}_1 = \mathbf{T}\ \mathbf{p} = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ 1 \end{bmatrix}$$

With this matrix, a cube is transformed in a parallelepiped.

Example

Similarly, the transformation

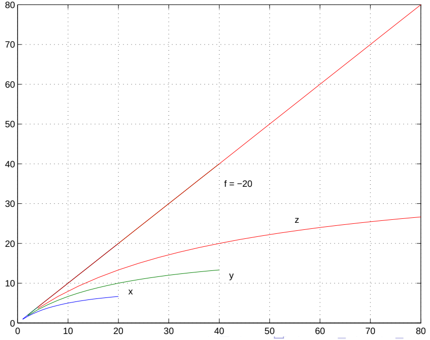
$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\frac{1}{f} & 0 & 1 \end{bmatrix}$$

performs a perspective transformation along **y**. The coordinates x, y, z of a point **p** are transformed in

$$x' = \frac{x}{1 - y/f}$$

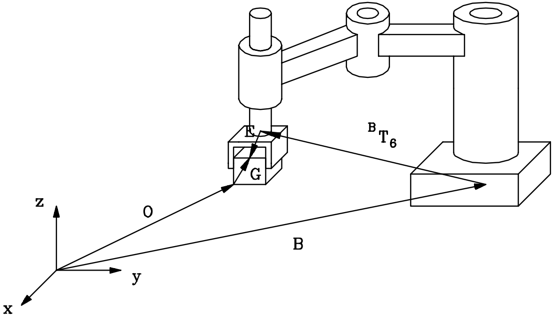
$$y' = \frac{y}{1 - y/f}$$

$$z' = \frac{z}{1 - y/f}$$



Equations with homogeneous transformations

Usually, in robotics it is necessary to specify the position/orientation of an object with respect to different reference frames (e.g. wrt to the end-effector, to the base frame, to other machines/tools, . . .).

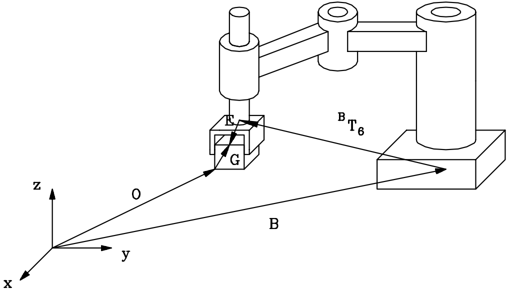


The following equation must be verified if the robot has to grasp the object:

$$\mathbf{B}\ {}^{B}\mathbf{T}_6\mathbf{E} = \mathbf{OG}$$

Equations with homogeneous transformations

$$\mathbf{B}\ {}^{B}\mathbf{T}_6\mathbf{E} = \mathbf{O}\mathbf{G}$$



Usually, matrices $\mathbf{B}, \mathbf{O}, \mathbf{G}, \mathbf{E}$ are known (and constant). Therefore, the equation can be solved in terms of ${}^{B}\mathbf{T}_6$

$${}^{B}\mathbf{T}_6 = \mathbf{B}^{-1}\mathbf{O}\mathbf{G}\mathbf{E}^{-1}$$

The robot configuration is then obtained.
Otherwise, the object position $\mathbf{O}$ (if not known) can be computed as

$$\mathbf{O} = \mathbf{B}\ {}^{B}\mathbf{T}_6\mathbf{E}\mathbf{G}^{-1}$$