Trajectory Planning for Robot Manipulators
Part 2

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Scaling trajectories

Due to several reasons, like limits on the actuation system (torques, accelerations, velocities, ...) or computational efficiency, it is often requested to *scale* trajectories and motion laws.

It is possible to adopt

- **Kinematic** scaling procedures
- **Dynamic** scaling procedures

Kinematic scaling of trajectories

If a trajectory is expressed in parametric form as a function of a parameter $\sigma = \sigma(t)$, by changing the parameterization it is possible to obtain in a simple manner a trajectory satisfying constraints on velocity or accelerations.

For this purpose, it is convenient to express the trajectory in *normal form*, i.e.:

$$p(t) = p_0 + (p_1 - p_0)s(\tau) = p_0 + Ls(\tau)$$

being $s(\tau)$ a proper parameterization, with

$$0 \leq s \leq 1, \qquad \tau = \frac{t - t_0}{t_1 - t_0} = \frac{t - t_0}{T}$$

In this manner, it results

$$\frac{dp}{dt} = \frac{L}{T} s'(\tau) \qquad \frac{d^2p}{dt^2} = \frac{L}{T^2} s''(\tau)$$

$$\frac{d^3p}{dt^3} = \frac{L}{T^3} s'''(\tau) \qquad \dots$$

$$\frac{d^np}{dt^n} = \frac{L}{T^n} s^{(n)}(\tau)$$

Kinematic scaling of trajectories

From

$$\frac{dp}{dt} = \frac{L}{T}s'(\tau) \qquad \frac{d^2p}{dt^2} = \frac{L}{T^2}s''(\tau)$$

$$\frac{d^3p}{dt^3} = \frac{L}{T^3}s'''(\tau) \qquad \ldots$$

$$\frac{d^np}{dt^n} = \frac{L}{T^n}s^{(n)}(\tau)$$

it follows that the maximum values for the velocity, acceleration, etc. are obtained in correspondence of the maximum values of the functions $s', s'', \ldots$.

These values and the corresponding time instants τ (t) are known from the chosen parameterization $s(\tau)$.

Notice that if the duration T of the trajectory is changed, it is possible to satisfy in an exact manner the given constraints or to optimize the trajectory itself (minimum time). Moreover, it is easily possible to co-ordinate more motion axes.

Kinematic scaling of trajectories

Polynomial trajectories of degree 3

Consider a parameterization expressed by a cubic polynomial

$$s(\tau) = a_0 + a_1\tau + a_2\tau^2 + a_3\tau^3, \qquad 0 \le s \le 1, \quad 0 \le \tau \le 1, \quad \tau = \frac{t}{T}$$

If the boundary conditions $p_0 = 0, v_0 = 0, v_1 = 0$ are specified, one obtains

$$a_0 = 0, \quad a_1 = 0, \quad a_2 = 3, \quad a_3 = -2$$

Therefore:

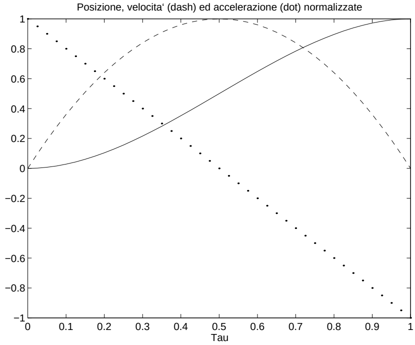
$$\begin{aligned} s(\tau) &= 3\tau^2 - 2\tau^3 \\ s'(\tau) &= 6\tau - 6\tau^2 \\ s''(\tau) &= 6 - 12\tau \\ s'''(\tau) &= -12 \end{aligned}$$

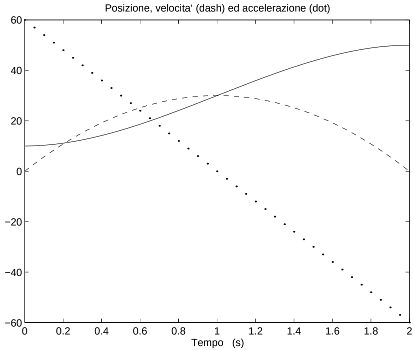
Kinematic scaling of trajectories

Then

$$s'_{max} = s'(0.5) = \frac{3}{2} \quad \Longrightarrow \quad \dot{q}_{max} = \frac{3L}{2T}$$

$$s''_{max} = s''(0) = 6 \quad \Longrightarrow \quad \ddot{q}_{max} = \frac{6L}{T^2}$$





Kinematic scaling of trajectories

Polynomial trajectories of degree 5

The polynomial $s(\tau)$ in normal form is now:

$$s(\tau) = a_0 + a_1\tau + a_2\tau^2 + a_3\tau^3 + a_4\tau^4 + a_5\tau^5, \qquad 0 \leq s \leq 1, \quad 0 \leq \tau \leq 1, \quad \tau$$

With null boundary conditions on accelerations and velocities, the following values for the parameters are obtained (trajectory 3-4-5)

$$a_0 = 0, \quad a_1 = 0, \quad a_2 = 0, \quad a_3 = 10 \quad a_4 = -14, \quad a_5 = 6$$

Then

$$\begin{aligned} s(\tau) &= 10\tau^3 - 15\tau^4 + 6\tau^5 \\ s'(\tau) &= 30\tau^2 - 60\tau^3 + 30\tau^4 \\ s''(\tau) &= 60\tau - 180\tau^2 + 120\tau^3 \\ s'''(\tau) &= 60 - 360\tau + 360\tau^2 \end{aligned}$$

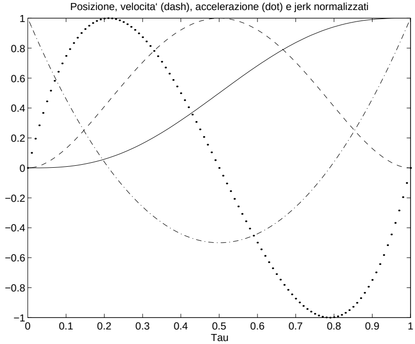
Kinematic scaling of trajectories

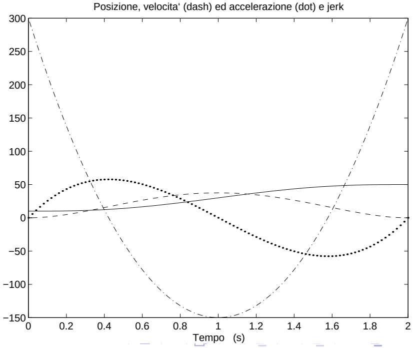
Therefore

$$s'_{max} = s'(0.5) = \frac{15}{8} \qquad \Longrightarrow \qquad \dot{q}_{max} = \frac{15L}{8T}$$

$$s''_{max} = s''(0.2123) = \frac{10\sqrt{3}}{3} \qquad \Longrightarrow \qquad \ddot{q}_{max} = \frac{10\sqrt{3}L}{3T^2}$$

$$s'''_{max} = s'''(0) = 60 \qquad \Longrightarrow \qquad \ddot{q}_{max} = 60\frac{L}{T^3}$$





Kinematic scaling of trajectories

Polynomial trajectories of degree 7

If a continuos jerk profile is requested, a polynomial with higher degree must be adopted. The normal form for a polynomial $s(\tau)$ of degree 7 is:

$$s(\tau) = a_0 + a_1\tau + a_2\tau^2 + a_3\tau^3 + a_4\tau^4 + a_5\tau^5 + a_6\tau^6 + a_7\tau^7$$

If null boundary conditions on velocity, acceleration and jerk are specified, the following parameters are obtained (trajectory 4-5-6-7)

$$a_0 = 0,\ a_1 = 0,\ a_2 = 0,\ a_3 = 0\ a_4 = 35,\ a_5 = -84,\ a_6 = 70,\ a_7 = -20$$

Therefore

$$\begin{aligned}
s(\tau) &= 35\tau^4 - 84\tau^5 + 70\tau^6 - 20\tau^7 \\
s'(\tau) &= 140\tau^3 - 420\tau^4 + 420\tau^5 - 140\tau^6 \\
s''(\tau) &= 420\tau^2 - 1680\tau^3 + 2100\tau^4 - 840\tau^5 \\
s'''(\tau) &= 840\tau - 5040\tau^2 + 8400\tau^3 - 4200\tau^4
\end{aligned}$$

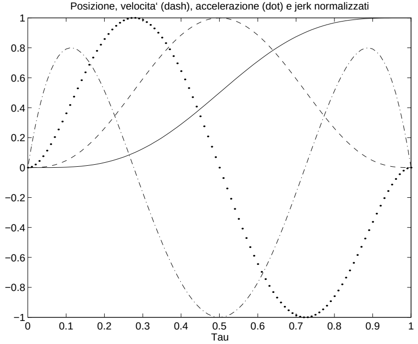
Kinematic scaling of trajectories

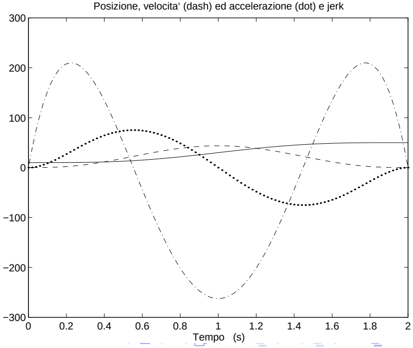
The maximum velocity and acceleration values are obtained for

$$s'_{max} = s'(0.5) = \frac{35}{16} \qquad \Longrightarrow \qquad \dot{q}_{max} = \frac{35L}{16\,T}$$

$$s''_{max} = s''\left(\frac{5 \pm \sqrt{5}}{10}\right) = \frac{84\sqrt{5}}{25} \qquad \Longrightarrow \qquad \ddot{q}_{max} = \frac{84\sqrt{5}}{25}\frac{L}{T^2}$$

$$s'''_{max} = s'''\left(\frac{1+\sqrt{3/5}}{2}\right) = 42, \qquad s'''_{min} = s'''(0.5) = -\frac{105}{2} \qquad \Longrightarrow \qquad \max_{\tau}|s'''| = \frac{105}{2}$$





Considerations on limits and durations of trajectories

From the previous examples, it is clear that if the displacement L and the duration T of a motion are specified, the profiles of velocity, acceleration and jerk are defined by the parameterization $s(\tau)$ chosen to generate the motion profile.
In particular, the maximum values for these variables are determined (for the sake of simplicity, consider the case $L > 0$).

	Pol. 3	Pol. 5	Pol. 7	Cicl.	Harmon.
Vel. $(*L/T)$	$\frac{3}{2} = 1.5$	$\frac{15}{8} = 1.875$	$\frac{35}{16} = 2.1875$	2	$\frac{\pi}{2} = 1.5708$
Acc. $(*L/T^2)$	6	$\frac{10\sqrt{3}}{3} = 5.7735$	$\frac{84\sqrt{5}}{25} = 7.5132$	$2\pi = 6.2832$	$\frac{\pi^2}{2} = 4.9348$
Jerk $(*L/T^3)$	12	60	$\frac{105}{2} = 52.25$	$4\pi^2 = 39.4784$	$\frac{\pi^3}{2} = 15.5031$

Notice that the polynomial of degree 7, originating a very smooth profile, requires higher velocity and acceleration values. Viceversa, the harmonic trajectory has a very good behavior.

Example: scaling a trajectory

Trajectory 3-4-5. Polynomial in *normal form*:

$$s(\tau) = a\tau^5 + b\tau^4 + c\tau^3 + d\tau^2 + e\tau + f$$

with

$$0 \leq s \leq 1, \qquad 0 \leq \tau \leq 1, \qquad \tau = \frac{t}{T}$$

The trajectory is

$$q(t) \;=\; q_0 + (q_1 - q_0)s(\tau) \;=\; q_0 + Ls(\tau)$$

and

$$\begin{aligned}
\dot{q}(t) &= Ls'(\tau)\frac{1}{T} \\
\ddot{q}(t) &= Ls''(\tau)\frac{1}{T^2} \\
&\dots \\
\frac{d^n q}{dt^n} &= Ls^{(n)}(\tau)\frac{1}{T^n}
\end{aligned}$$

Example: scaling a trajectory

Then (trajectory 3-4-5, $f = e = d = 0$, and $a = 6, b = -15, c = 10$)

$$\begin{aligned} s'(\tau) &= 30\tau^4 - 60\tau^3 + 30\tau^2 \\ s''(\tau) &= 120\tau^3 - 180\tau^2 + 60\tau \\ s'''(\tau) &= 360\tau^2 - 360\tau + 60 \end{aligned}$$

and

$$s'_{max} = s'(0.5) = \frac{15}{8} \qquad \Longrightarrow \qquad \dot{q}_{max} = \frac{15L}{8T}$$

$$s''_{max} = s''(0.2123) = \frac{10\sqrt{3}}{3} \qquad \Longrightarrow \qquad \ddot{q}_{max} = \frac{10\sqrt{3}L}{3T^2}$$

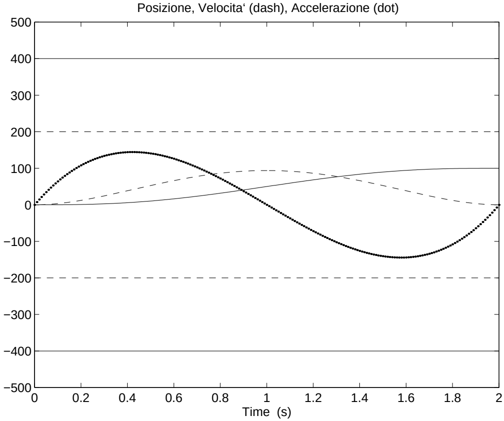
$$\ldots$$

Given constraints on maximum acceleration and velocity, it is possible to properly scaling the trajectory.

Co-ordination of more motion axes made on the basis of the “most stressed” actuator.

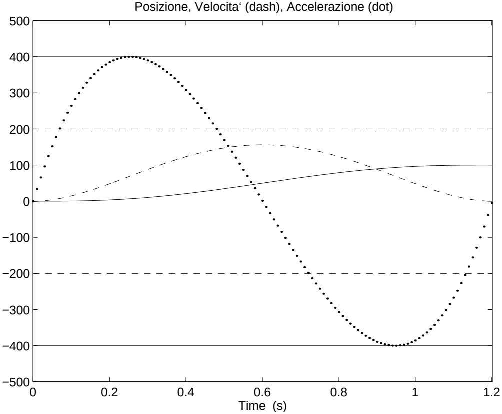
Example: scaling a trajectory

If: $q_0 = 0;\qquad q_1 = 100;\qquad t_0 = 0;\qquad t_1 = 2;\qquad \dot{q}_{max} = 200;\qquad \ddot{q}_{max} = 400$



Example: scaling a trajectory

$$T_{min,v} = \frac{15L}{8\dot{q}_{max}} = 0.9375\ s, \qquad T_{min,a} = \sqrt{\frac{10\sqrt{3}L}{3\ddot{q}_{max}}} = 1.2014\ s \qquad T_{min} = \max\{T_{min,v},\ T_{min,a}\}$$



Dynamic scaling of trajectories

When a trajectory is specified for a complex mechanical system, because of the dynamics of the actuation system, of the robot manipulator or of the load (dynamic couplings), torques non physically achievable by the actuators could be requested. In these cases, it is possible to scale the trajectory taking into account the dynamics of the system in order to obtain a physically achievable motion.
The dynamic model of a manipulator is

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \boldsymbol{\tau}$$

Then, for each joint

$$\mathbf{m}_i^T(\mathbf{q})\ddot{\mathbf{q}} + \frac{1}{2}\dot{\mathbf{q}}^T\mathbf{C}_i(\mathbf{q})\dot{\mathbf{q}} + g_i(\mathbf{q}) = \tau_i \qquad i = 1, \ldots, n$$

If

$$\mathbf{q} = \mathbf{q}(\sigma) \qquad \sigma = \sigma(t)$$

is a proper parameterization of the trajectory with a motion law such that

$$\dot{\mathbf{q}} = \frac{d\ \mathbf{q}}{d\sigma}\dot{\sigma}, \qquad \ddot{\mathbf{q}} = \frac{d^2\ \mathbf{q}}{d\sigma^2}\dot{\sigma}^2 + \frac{d\ \mathbf{q}}{d\sigma}\ddot{\sigma}$$

Dynamic scaling of trajectories

By substitution in the dynamic model:

$$\left[m_i^T(\mathbf{q}(\sigma)) \frac{d\ \mathbf{q}}{d\sigma} \right] \ddot{\sigma} + \left[m_i^T(\mathbf{q}(\sigma)) \frac{d^2\ \mathbf{q}}{d\sigma^2} + \frac{1}{2} \frac{d\ \mathbf{q}}{d\sigma}^T \mathbf{C}_i(\mathbf{q}(\sigma)) \frac{d\ \mathbf{q}}{d\sigma} \right] \dot{\sigma}^2 + g_i(\mathbf{q}(\sigma)) = \tau_i$$

from which

$$\alpha_i(\sigma)\ddot{\sigma} + \beta_i(\sigma)\dot{\sigma}^2 + \gamma_i(\sigma) = \tau_i$$

Notice that $\gamma_i(\sigma)$ (gravitational terms) depend on the position only (σ).

Dynamic scaling of trajectories

Let us suppose to compute the torques τ_i necessary to achieve the motion defined by $\mathbf{q} = \mathbf{q}(\sigma),\ \sigma = \sigma(t)$:

$$\tau_i(t) = \alpha_i(\sigma(t))\ddot{\sigma}(t) + \beta_i(\sigma(t))\dot{\sigma}^2(t) + \gamma_i(\sigma(t)), \qquad i = 1, ..., n, \qquad t \in [0,\ T]$$

If the time-axis is changed (e.g. in a linear fashion ($x = kt$)), a different parameterization of the trajectory is obtained

$$t \rightarrow x = kt \qquad x \in [0,\ kT] \qquad \sigma(t) \rightarrow \hat{\sigma}(x)$$

Notice that in general even a non linear parameterization $x = x(t)$ could be considered.

With the new parameterization, one obtains:

$$\begin{aligned} \hat{\sigma}(x) &= \sigma(t) \\ \dot{\hat{\sigma}}(x) &= \frac{\dot{\sigma}(t)}{k} \\ \ddot{\hat{\sigma}}(x) &= \frac{\ddot{\sigma}(t)}{k^2} \end{aligned}$$

Dynamic scaling of trajectories

Therefore

- if $k > 1$ a *slower* motion is obtained
- if $k < 1$ a *faster* motion is obtained.

With the new parameterization, the torques compute as:

$$\begin{aligned} \tau_i(x) &= \alpha_i(\hat{\sigma}(x))\ddot{\hat{\sigma}}(x) + \beta_i(\hat{\sigma}(x))\dot{\hat{\sigma}}^2(x) + \gamma_i(\hat{\sigma}(x)) \\ &= \alpha_i(\sigma(t))\frac{\ddot{\sigma}(t)}{k^2} + \beta_i(\sigma(t))\frac{\dot{\sigma}^2(t)}{k^2} + \gamma_i(\sigma(t)) \\ &= \frac{1}{k^2}[\tau_i(t) - \gamma_i(\sigma(t))] + \gamma_i(\sigma(t)) \end{aligned}$$

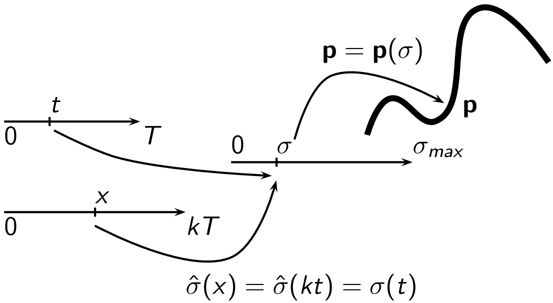
from which

$$\tau_i(x) - \gamma_i(x) = \frac{1}{k^2}[\tau_i(t) - \gamma_i(t)]$$

Dynamic scaling of trajectories

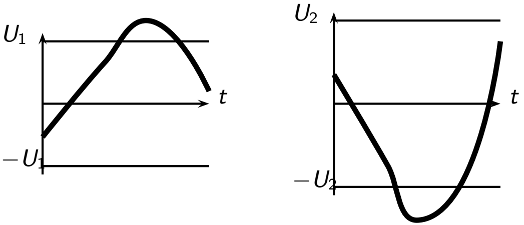
Some considerations:

- it is not necessary to re-compute the whole trajectory
- neglecting the gravitational term, the ‘new’ torques are obtained by scaling by the factor $1/k^2$ the ‘old’ torques.
- the motion os slower if $k > 1$, and it is faster if $k < 1$ (total duration equal to kT)



Dynamic scaling of trajectories

Example: Let consider a 2 dof manipulator, for which the following torques should be generated:



Defining:

$$k^2 = max \left\{ 1, \frac{|\tau_1|}{U_1}, \frac{|\tau_2|}{U_2} \right\} \geq 1$$

then:

- $x = kt$
- total time $= kT \geq T$

Then, the new torques are physically achievable, $(\tau(x) = \tau(t)/k^2)$, and at least one of them saturates in a point.

A *variable scaling* can be adopted to avoid slowing down the whole trajectory (saturation usually occurs in a single point).

For the optimal motion law (minimum time), at least one actuator saturates in each period.