Trajectory Planning for Robot Manipulators
Part 4

Claudio Melchiorri

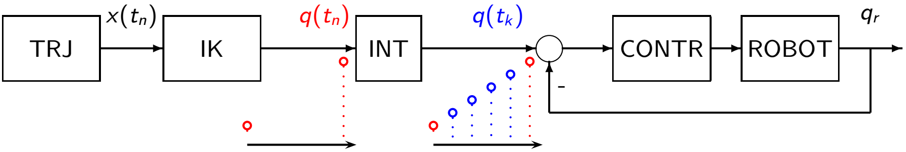
Dipartimento di Elettronica, Informatica e Sistemistica (DEIS)

Università di Bologna

email: claudio.melchiorri@unibo.it

Workspace trajectories

If trajectories are defined in the workspace, it is necessary to use the inverse kinematic function to translate the motion specification to the joint space (where actuators operate). Since this increases the computational burden for trajectory planning, the operations of computing the trajectory and translating it to the joint space are made at a lower frequency with respect to the control frequency. Therefore, it is necessary to interpolate the data before assigning them to the low-level controllers: usually, a simple linear interpolation is adopted.



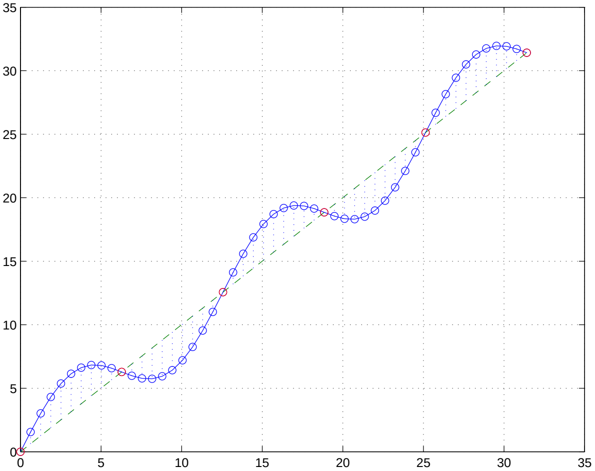
Typical values: $\Delta t_n = 10$ ms, $\Delta t_l = 1$ ms

$\Longrightarrow$ 10 values of $q(t_k)$ for each value of $q(t_n)$ $(x(t_n))$

$\Longrightarrow$ there is a delay Δt_n between the two sequences $q(t_k)$ and $q(t_n)$

Workspace trajectories

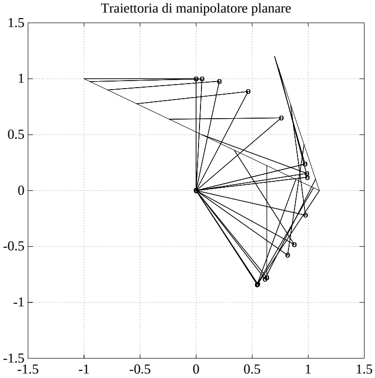
Another problem: the Cartesian positions actually achieved during the motion obtained by interpolating the points $q(t_n)$ are not those originally planned.



Workspace trajectories

For the computation of the workspace trajectories, it is possible to adopt one of the techniques used for the joint space (substituting the joint variable $q(t)$ with $x(t)$, i.e. a position or an orientation in the Cartesian space) or to define analytically the geometric path (e.g. an ellipse) as a function of time (i.e. $p = p(t)$) or, better, in a parametric form $\mathbf{p} = \mathbf{p}(s)$, being $s = s(t)$ a proper parameterization defining the motion law.

Example: planar 2 dof manipulator



Desired trajectory:

- total duration 3s,
- start in $p_i = [-1.0,\ 1.0]$
- end in $p_f = [0.7,\ 1.2]$
- composed by two linear segments with intermediate point $p_m = [1.1,\ 0.0]$ per $t_m = 2s$.

Example: planar 2 dof manipulator

Consider the parametric form

$$x(t) = x_0 + \frac{x_1 - x_0}{\Delta T}\,\tau$$
$$y(t) = y_0 + \frac{y_1 - y_0}{\Delta T}\,\tau$$

where $\Delta T = t_1 - t_0$, and $\tau \in [0, \Delta T]$ is defined so that desired position/velocity profiles are obtained, for example linear segments with parabolic blends (position in the workspace). The kinematic model is

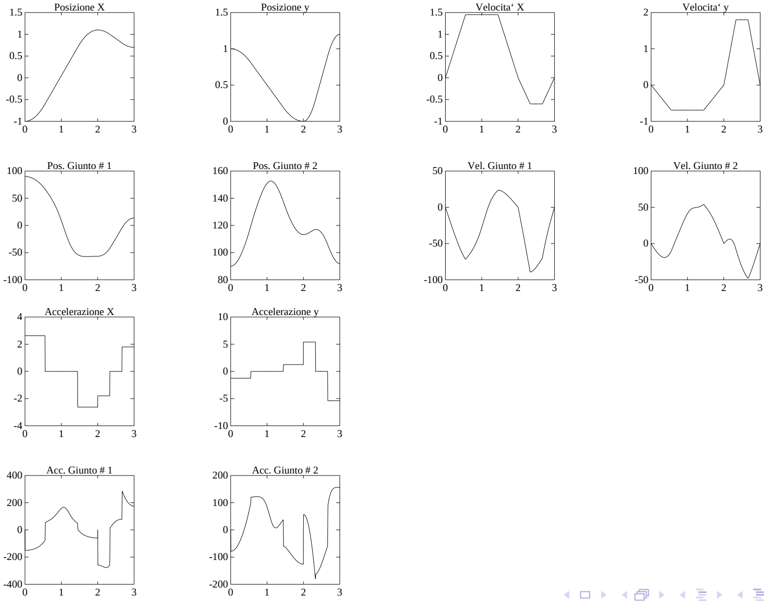
$$x = l_1 C_1 + l_2 C_{12} \qquad\qquad y = l_1 S_1 + l_2 S_{12}$$

while the inverse kinematic equations are

$$C_2 = \frac{x^2 + y^2 - a_2^2 - a_3^2}{2 a_2 a_3} \qquad\qquad S_2 = \sqrt{1 - C_2^2}$$

$$\theta_2 = \text{atan2}\,(S_2,\; C_2) \qquad\qquad \theta_1 = \text{atan2}\,(y, x) - \text{atan2}\,(a_2 S_2,\; a_1 + a_2 C_2)$$

Example: planar 2 dof manipulator



Position trajectories

To plan a trajectory in the workspace, usually the geometric path p (line, circle, ellipse, ...) is defined as a function of a parameter $s(t)$: $p = p(s)$.

The parameter $s = s(t)$ is computed by using one of the techniques discussed for joint space trajectories. A classical approach is to plan $s(t)$ as a linear function with parabolic blends, in order to have in the work space acceleration/deceleration tracts (low stress for the mechanical and actuation system).

Notice that for parameterized trajectories the following conditions hold:

$$\dot{\mathbf{p}} = \frac{d\ \mathbf{p}}{ds}\dot{s}, \qquad \ddot{\mathbf{p}} = \frac{d\ \mathbf{p}}{ds}\ddot{s} + \frac{d^2\ \mathbf{p}}{ds^2}\dot{s}^2$$

Position trajectories

Curvature of a geometric path

Consider a path Γ in the workspace $\mathbb{R}^3$, expressed in parametric form

$$p = p(r) = \begin{bmatrix} x(r) \\ y(r) \\ z(r) \end{bmatrix}, \qquad r \in [r_a, \quad r_b]$$

Assume that the curve is *regular*, i.e.

$$\dot{p} = \frac{d\ p}{dr} \neq 0, \qquad \forall r \in [r_a, \quad r_b]$$

Given a point p_a of Γ, and a motion direction on the path, the *arc lenght* of a generic point $p(r)$ is defined as

$$s = \int_{p_a}^{p(r)} ||\dot{p}(\rho)|| d\rho$$

Position trajectories

By definition, the arc length represents the length of the arc of Γ defined by the two points p and p_a (if p follows p_a, or the opposite of such a length if p is before p_a). The value $s = 0$ is assigned to point p_a.
A bijective relationship exists between the values of the arc length s and the points of the path Γ, and then it is possible to use the arc length for a parametric expression of Γ.

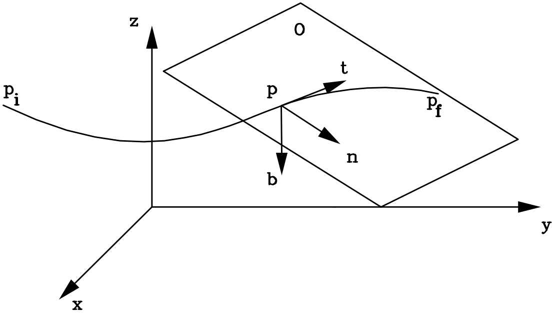
$$p = p(s)$$

It is possible to assign to each point p of Γ a reference frame (Frenet frame) defined by the following unit vectors

$$\begin{cases} \mathbf{t} = \dfrac{\dot{\mathbf{p}}}{||\dot{\mathbf{p}}||} & \text{tangent unit vector} \\ \mathbf{b} = \dfrac{\dot{\mathbf{p}} \times \ddot{\mathbf{p}}}{||\dot{\mathbf{p}} \times \ddot{\mathbf{p}}||} & \text{binormal unit vector} \\ \mathbf{n} = \mathbf{b} \times \mathbf{t} & \text{normal unit vector} \end{cases}$$

Position trajectories

- The unit vector $\mathbf{t}$ lies along the direction tangent to Γ in $\mathbf{p}$, and is directed along the positive s direction
- The unit vector $\mathbf{n}$ defines, with $\mathbf{t}$, the *osculating plane O*, defined as the plane containing point $\mathbf{p}$ and a point $\mathbf{p}' \in \Gamma$ when $\mathbf{p}' \to \mathbf{p}$.
- The unit vector $\mathbf{b}$ (*binormal*) is defined so that the frame $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ is right-handed. Notice that it is not always possible to define uniquely the Frenet frame.



Position trajectories

Segment of a line

The linear geometric path between points $\mathbf{p}_i$ and $\mathbf{p}_f$ has a parametric representation expressed by

$$\mathbf{p}(s) = \mathbf{p}_i + \frac{s}{||\mathbf{p}_f - \mathbf{p}_i||}(\mathbf{p}_f - \mathbf{p}_i), \qquad s \in [0, \; ||\mathbf{p}_f - \mathbf{p}_i||]$$

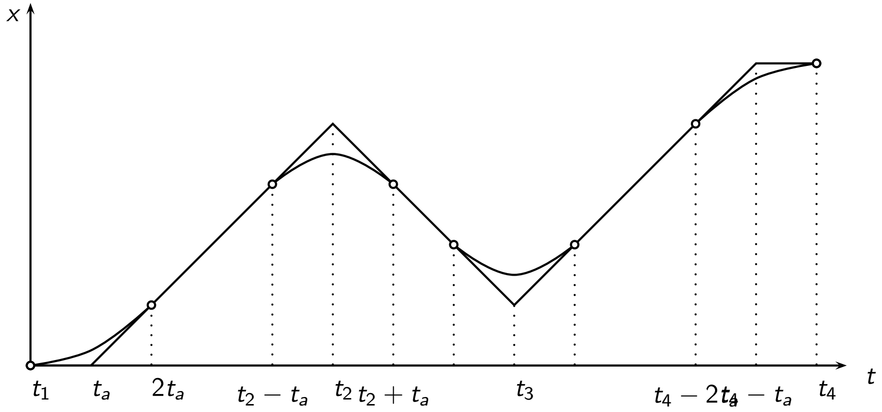
Moreover, by deriving $\mathbf{p}$ with respect to s, one obtains

$$\frac{d\;\mathbf{p}}{ds} = \frac{\mathbf{p}_f - \mathbf{p}_i}{||\mathbf{p}_f - \mathbf{p}_i||}, \qquad \frac{d^2\;\mathbf{p}}{ds^2} = 0$$

It is possible to plan a trajectory through a sequence of points with the same modalities seen in the joint space. If it is required to pass exactly through the intermediate points, then it is possible to compute the parameter s using one of the motion laws defined in the joint space (e.g. cubic, trapezoidal, ...).
In case it is not required for the manipulator to pass through the intermediate points, the geometric path can be defined for example by linear segments with polynomial blends (position error, but non null velocity in the via points).

Position trajectories

A typical profile is shown below. The variable x is defined with a sequence of points interpolated with linear segments, while the real trajectory only approximates (in the vicinity of the via points) the given path.



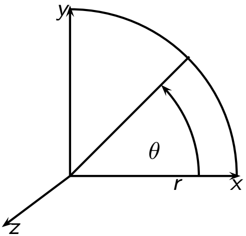
Position trajectories

Arc of a circle

A parametric representation of an arc of a circle is

$$\mathbf{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r\cos(\theta) \\ r\sin(\theta) \\ 0 \end{bmatrix} \qquad \theta \in [\theta_{min}, \theta_{max}]$$

where the parameter is the angle $\theta = \theta(t)$. Notice that if the path must be arbitrarily positioned/oriented in the 3D space, it is sufficient to multiply the (homogeneous) vector $\mathbf{p}$ by a proper transformation matrix T.

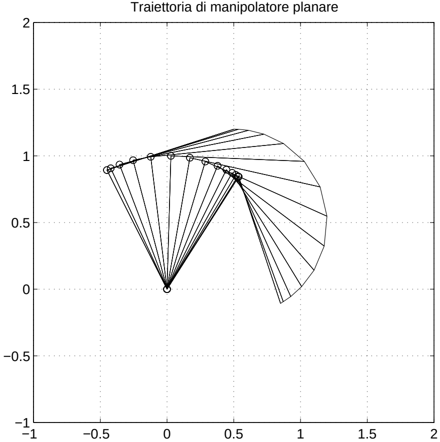


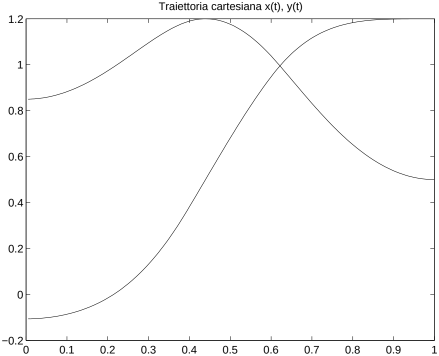
A motion law with acceleration/deceleration tracts (in the operational space) is obtained if the parameter (in this case: θ) is computed, for example, with a trapezoidal velocity profile.

Position trajectories - Example

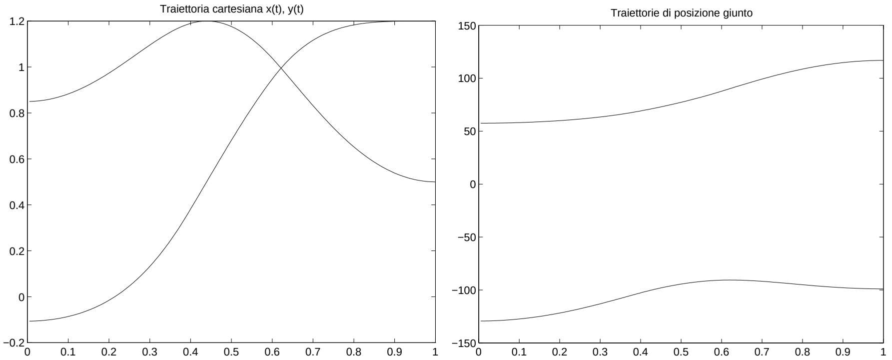
Planar 2 dof manipulator with links' lenght: $a_1 = a_2 = 1$. The desired circular motion is defined by

$$\text{center} = [0.5,\, 0.5], \qquad \text{radius r} = 0.7$$
$$\theta_i = -60, \quad \theta_f = 90, \quad \text{T} = 1 \text{ s}$$





Position trajectories



Rotational trajectories

Planning trajectories in terms of changes in orientation is somehow more complex than planning in position only. While it is quite simple to plan a motion between points $\mathbf{p}_i$ and $\mathbf{p}_f$, the same is not true for interpolating the orientation between two rotational matrices $\mathbf{R}_i$ and $\mathbf{R}_f$: for example if the elements r_{ij} are changed linearly from the initial (in $\mathbf{R}_i$) to the final (in $\mathbf{R}_f$) value, there is not guarantee that the intermediate matrices are real rotation matrices (orthogonal columns with unit norm).

Usually, the Euler or RPY angles are employed or, alternatively, the angle/axis representation.

With the Euler or RPY angles, two triples ϕ_i, ϕ_f are defined, and an interpolation based on one of the presented techniques can be adopted (advisable in any case continuity at least of the in rotational velocity).

Rotational trajectories

With the angle/axis representation, if $\mathbf{R}_i$ and $\mathbf{R}_f$ are the initial and final rotation matrices, then a matrix $\mathbf{R}_{i,f}$ exists such that

$$\mathbf{R}_i \ \mathbf{R}_{i,f} = \mathbf{R}_f$$

or

$$\mathbf{R}_{i,f} = \mathbf{R}_i^T \mathbf{R}_f = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

Then, the unit vector $\mathbf{w}$ and the rotational angle θ are

$$\theta_r = \text{acos} \frac{r_{11} + r_{22} + r_{33} - 1}{2} \qquad (1)$$

$$\mathbf{w} = \frac{1}{2 \sin \theta_r} \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix} \qquad (2)$$

Rotational trajectories

It is now necessary to define a matrix $\mathbf{R}_t(t)$ so that $\mathbf{R}_t(0) = \mathbf{I}$ and $\mathbf{R}_t(t_f) = \mathbf{R}_{i,f}$.
A choice can be

$$\mathbf{R} = \begin{bmatrix} w_x^2(1-C_\theta)+C_\theta & w_x w_y(1-C_\theta)-w_z S_\theta & w_x w_z(1-C_\theta)+w_y S_\theta \\ w_x w_y(1-C_\theta)+w_z S_\theta & w_y^2(1-C_\theta)+C_\theta & w_y w_z(1-C_\theta)-w_x S_\theta \\ w_x w_z(1-C_\theta)-w_y S_\theta & w_y w_z(1-C_\theta)+w_x S_\theta & w_z^2(1-C_\theta)+C_\theta \end{bmatrix}$$

where $\theta(t)$ is computed according to one of the previous motion law (cubic, trapezoidal, ...) from $\theta(0) = 0$ to $\theta(t_f) = \theta_r$, while $\mathbf{w}$ is defined as in (2).

The following rotation matrix is then obtained

$$\mathbf{R}(t) = \mathbf{R}_i \mathbf{R}_t(\theta(t))$$

Workspace trajectories

Final considerations

Some techniques for planning trajectories in the joint and in the work space have been illustrated.

If the trajectory is planned in the work space, the end-effector moves along well defined paths, a very important aspect in many industrial applications.

On the other hand, the computational burden is higher in case of work-space trajectories. For this reason, the frequency at which the trajectory in computed in lower than the control frequency, and an interpolation is then necessary.

Moreover, since the velocity/acceleration/torque limits required in the work-space may result non physically achievable in the joint space (i.e. in the actuation space) a re-computation of the trajectory might be necessary.

Workspace trajectories

Final considerations

Finally, singular configurations may generate problems if the trajectory is planned in the work space.

As a matter of fact, if a motion defined in the work space reaches points close to singular configuration, it should be avoided. Therefore, the trajectory should be checked in advance and, in case, not actuated or modified.

Clearly all these problems are not present if the trajectory is planned in the joint space.