Nonlinear Robust Control of a Reduced-Complexity Ducted MAV for Trajectory Tracking

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Abstract—In this paper we address the problem of controlling the vertical, lateral, longitudinal and yaw attitude motion of a nonlinear model of a ducted-fan Micro Aerial Vehicle (MAV) of reduced mechanical complexity. The references are assumed to be arbitrary signals with only some limitations in the higher order time derivatives imposed by functional controllability and actuator constraints. The nonlinear control law proposed succeeds in enforcing the desired trajectories robustly with respect to uncertainties characterizing the physical and aerodynamical parameters of the ducted-fan MAV.

I. INTRODUCTION

Goal of this paper is to present preliminary results framed within a broader project regarding the development of a small size UAV (Unmanned Aerial Vehicle), namely an aerial vehicle able of fully autonomous or partially-supervised remote controlled flight. The final purpose is to design and build an innovative VTOL (Vertical Take-off and Landing) UAV, suitable for a large number of civil applications, such as mobile video-surveillance, forest fire detection and surveillance, environmental pollution monitoring, detailed relief map survey in un-accessible sites, traffic monitoring etc. The starting design idea is to develop a "tail-sitter" UAV able to vertically take-off/land and changes its attitude during different flight phases in order to optimize mission accomplishment.

Traditionally, the most important architecture of Rotary Wing Unmanned Aerial Vehicle (RWUAV) is represented by the helicopter whose mechanical structure relies upon a main rotor generating the main thrust and a tail rotor used to compensate for main rotor torque. Full controllability along the lateral, longitudinal and vertical direction is given by the collective and cyclic pitch command which, in turn, are responsible of the high mechanical complexity of this kind of vehicles and, indeed, of their scarce reliability. Furthermore mechanical complexity of this kind of vehicles represents also an obstacle to size reduction and limit the applicability of helicopter architectures to design MAV.

To overcome these intrinsic limitations we study a simple configuration of a ducted-fan rotary-wing MAV which could be seen as composed by two main parts, the first one constituted by a single fixed pitch rotor, generating the main thrust, and the last one constituted by some actuated aerodynamic surfaces whose effect, in conjunction with the main rotor flux, is to provide the force and torque components needed to acquire full controllability. The effectiveness of similar UAV configurations is testified by recent contributions (see for example [2], [9] and [8]) presenting preliminary results and tests along this design direction. Differently from ([8]) we do not allow the presence of counter rotating propellers in order to overcome gyroscopic precession effects on the rigid body dynamics.

In this paper we concentrate on a possible architecture, deriving the associated dynamical model, and we design a non linear robust regulator capable of asymptotic tracking performances, provided some restrictions on higher order references time derivatives. To this end we take advantage of the design techniques proposed in [7] (see also [5] and [4]) developed for helicopters using a mix of saturated and high gain feedback and feedforward control to achieve tracking objectives. In particular we show how a modification of the control structure proposed in [7] succeeds in asymptotically tracking the references provided that the latters fulfill specific bounds on the higher order time derivatives as better specified in the paper. Future research efforts will aim test architectural solutions and control algorithms on prototypes using the necessary avionics.

Notations For a bounded function \( s : \mathbb{R} \rightarrow \mathbb{R}^r \) we denote \( \|s\|_{\infty} = \sup_{t \geq 0} \|s(t)\| \) and \( \|s\|_a = \lim_{t \rightarrow \infty} \sup \|s(t)\| \) in which \( \|\cdot\| \) denotes the Euclidean norm. We use the compact notation \( C_a, S_a, T_a \) with \( a \in \mathbb{R} \) to indicate respectively \( \cos a, \sin a \) and \( \tan a \). For a vector \( \omega = (\omega_1, \omega_2, \omega_3)^T \), 

Skew(\( \omega \)) denoted the \( 3 \times 3 \) skew-symmetric matrix with the first, second and third row respectively given by \([0, -\omega_3, \omega_2], [\omega_3, 0, -\omega_1]\) and \([\omega_2, \omega_1, 0]\).

II. RIGID BODY DYNAMICS

The architecture of ducted fan MAV considered in this paper is shown figure 1 and can be thought as divided into two different subsystems. The first one is given by a fixed pitch rotor driven by an engine. This subsystem generates the necessary thrust in order to actuate the overall system. The second subsystem is given by four controlled surfaces positioned below the main rotor deviating the air flow coming from the fan in order to compensate for engine torque and to generate the forces and torques necessary to control the system.
A mathematical model for the system can be derived from Newton-Euler equations of motion of a rigid body in the configuration space $SE(3) = \mathbb{R}^3 \times SO(3)$. In particular, by fixing an inertial coordinate frame $F_i$ and a coordinate frame $F_b$ attached to the body, the model of the RWUA V with respect to the inertia framework can be described as

$$
M \ddot{\mathbf{x}} = R f^b \\
J \ddot{\boldsymbol{\omega}} = -S\text{Skew}(w)Jw + w_c Gw + \tau^b
$$

where $f^b$ and $\tau^b$ represents respectively the vector of forces and torques applied to the vehicle expressed in the body frame, $M$ the vehicle total mass, $J = \text{diag}(J_x, J_y, J_z)$ the diagonal inertia matrix (where, due to the particular shape of the UAV, $J_x \equiv J_y$), the vector $w = \text{col}(x, y, z)$ the position of the center of mass, $w$ the angular velocity expressed in the body frame, $R$ the rotation matrix relating the two reference frames and $G$ is the skew symmetric matrix

$$
G = \begin{pmatrix}
0 & -I_{rot} & 0 \\
I_{rot} & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
$$

in which $I_{rot}$ is the inertia of the propeller along the spin axis (see [1]). The term $w_c Gw$ in (1) models the gyroscopic precession torque effect due to the angular speed $w_c$ of the fan.

The rotation matrix is expressed in terms of roll ($\phi$), pitch ($\theta$) and yaw ($\psi$) representation as

$$
R = \begin{pmatrix}
C_\phi C_\theta & -S_\phi S_\psi C_\theta + C_\phi S_\phi S_\psi & S_\phi S_\psi C_\theta + C_\phi S_\phi S_\psi \\
S_\phi C_\theta & C_\phi C_\psi + S_\phi S_\theta S_\psi & -C_\phi S_\psi + S_\phi S_\theta S_\psi \\
-S_\theta & C_\theta S_\phi & C_\theta C_\phi
\end{pmatrix}
$$

The attitude dynamics is then defined as

$$
\dot{\Theta}_\psi = Q(\Theta) \omega \\
Q(\Theta) = \begin{pmatrix}
1 & S_\phi T_\theta & C_\phi T_\theta \\
0 & C_\theta & -S_\theta \\
0 & S_\theta / C_\theta & C_\phi / C_\theta
\end{pmatrix}
$$

where, for convenience, we denote

$$
\Theta := \begin{pmatrix}
\phi \\
\theta \\
\psi
\end{pmatrix}, \quad \Theta_\psi := \begin{pmatrix}
\Theta \\
\psi
\end{pmatrix}
$$

To model force and torque generation mechanisms we make use of simple aerodynamics arguments considering the system in almost stationary flight. For a blade element representing a small wing with infinitesimal area $dS$ moving into air with relative wind velocity $V$, the lift and the drag forces (see [10]) are given respectively by

$$
dL = \frac{1}{2} \rho V^2 C_L dS \quad dD = \frac{1}{2} \rho V^2 C_D dS
$$

where $\rho$ is air density and $C_L, C_D$ are respectively the lift and drag coefficients. Assuming airfoil profiles with small Reynolds numbers and with reasonably small angles of attack, we make use of the following expressions

$$
C_L = K_L p \quad C_D = K_D p^2
$$

with $p$ the angle of attack with respect to relative wind and $K_L, K_D$ constant coefficients collecting other geometric parameters.

Similar arguments could be used to model ducted fan thrust $T$ and resistance torque $Q$ as

$$
T = K_T w_c^2 \quad Q = K_Q w_c^2
$$

with $K_T$ and $K_Q$ constant coefficients. Output velocity of the air generated by the main rotor, denoted by $V_r$, is assumed to be proportional to angular fan velocity $w_c$. In our model air velocity $V_r$ represents also the relative wind velocity to calculate forces generated by each active flap using (3).

To model those forces we refer to our design configuration shown in figure 2. First we will consider singularly all forces generated by each active flap and then we will consider the resultant contributions that affect the rigid body dynamics. Geometric pitch of each active flap is obtained as the sum of two different control inputs. The first one, denoted with $c$, is the same for all four surfaces and represent the control action able to generate the anti-torque necessary to counteract engine one and to obtain controllability of $\psi$ dynamics. Second one, $a$ or $b$ dependently on which flap is considered, is used to obtain two different forces directed respectively along $x$ and $y$ axis. These forces are used to govern both $\Theta$ and, by propeller thrust projection, lateral/longitudinal dynamics. Since all four surfaces are equivalent, we denote with $K_{FL}$ and $K_{FR}$ respectively the constant lift and drag coefficients of each flap. With an eye to figure 2, by means of equations (3) and (4), the expressions of the modules of lift forces $F_L^i$ and drag forces $F_D^i$ associated to each flap are given by

$$
F_L^1 = K_{FL} w_c^2 (c + b) \quad F_D^1 = K_{FD} w_c^2 (c + b)^2 \\
F_L^2 = K_{FL} w_c^2 (c - b) \quad F_D^2 = K_{FD} w_c^2 (c - b)^2 \\
F_L^3 = K_{FL} w_c^2 (c + a) \quad F_D^3 = K_{FD} w_c^2 (c + a)^2 \\
F_L^4 = K_{FL} w_c^2 (c - a) \quad F_D^4 = K_{FD} w_c^2 (c - a)^2
$$

whereas the direction of each $F_L^i$ and $F_D^i$ is shown in Figure 2. Consider first the overall effects of the lift forces and observe that the module of each $F_L^i$ is a linear function of the control inputs $a$, $b$ and $c$, i.e. we could consider separately the effects of each control inputs in the overall dynamics. Since the centers of pressure of any two opposite surfaces are at distance $d_T$, by means of control input $c$ we obtain a resultant torque $Q_F$ directed along $z$ axis and given by

$$
Q_F = -2 K_{FL} d_T w_c^3 c \quad \text{whereas inputs } a \text{ and } b \text{ are in charge}
$$
In summary, the external wrench vector \( \text{col}(f^b, \tau^b) \) applied to the rigid body can be seen as a nonlinear function of four control inputs

\[
u = \text{col}(\tau_e, c, a, b)
\]

In the following, to simplify the force generation mechanism, we assume that structurally the contribution of both drag forces \( D_F \) along body \( z \) axis and forces \( f_x \) and \( f_y \) along respectively body \( x \) and \( y \) axis are negligible with respect to the contribution of main thrust \( T \), rewriting the overall external wrench vector as

\[
f^b = w_c^2 \begin{pmatrix} 0 \\ 0 \\ -K_T \end{pmatrix} + R^T \begin{pmatrix} 0 \\ 0 \\ Mg \end{pmatrix}
\]

\[
\tau^b = A(w_c^2, c)\nu + B(\tau_e)
\]

with

\[
v = \text{col}(a, b, c)
\]

and with

\[
A(w_c^2, c) = \begin{pmatrix} 0 & A_1(w_c^2, c) \\ 0 & A_2(w_c^2) \end{pmatrix}
\]

\[
B(\tau_e) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

where \( g \) is the force of gravity and where

\[
A_1(w_c^2, c) = \begin{pmatrix} -t_{dc} & -t_1 \\ t_1 & t_{dc} \end{pmatrix}
\]

\[
A_2(w_c^2) = -t_x w_c^2
\]

Due to design characteristics the matrix \( A(w_c^2, c) \) is structurally full rank. This fact is a simple consequence of having efficient airfoil profiles, with \( K_{F_k} \gg K_{F_0} \), and, as a design constraint in order to achieve the desired controllability, distance \( d \) greater than the distance \( d_T \) so that \( ||t_{dc}|| < t_1 \) for all admissible values of the control input \( c \).

The overall system is described by (1), (2), (5), (7) and (8). It is 13th order model with the 4 control inputs (6).

\[
\begin{align*}
M &= M_0 + M_\Delta, & J &= J_0 + J_\Delta \\
K_T &= K_{T,0} + K_{T,\Delta}, & i \in \{M, Q\} \\
K_F &= K_{F,0} + K_{F,\Delta}, & i \in \{L, D\} \\
d &= d_0 + d_\Delta, & d_T &= d_{T,0} + d_{T,\Delta}
\end{align*}
\]

These uncertainties reflect into uncertainties of the matrix \( A(\cdot) \) introduced in (10) which will be accordingly written as

\[
A(w_c^2, c) = A_0(w_c^2, c) + A_\Delta(w_c^2, c)
\]
All the uncertain parameters are collected in a single uncertain
vector $\mu$. The ranges of the uncertainties of the physical
parameters will be not constrained to be "small" but will be
allowed in general to be "arbitrarily large" (fulfilling only
physical constraints). In particular we will assume that $\mu \in \mathcal{I}$, with $\mathcal{I}$ a known compact set. The only mild requirement
needed to support the results presented in this paper, is a
restriction on the relative variation of $A(\cdot)$ with respect to its
nominal value $A_0(\cdot)$. In particular it is required the existence
of a positive number $m^* < 1$ such that
$$
\|A_\Delta \left( w^*_e, c \right) A_0 \left( w^*_e, c \right)^{-1} \| \leq m^* I
$$
for all possible values of $(w_e, c)$ within physical ranges.

### III. Control Problem Overview

In this paper we address the problem of designing the four
control inputs (6) in order to asymptotically track vertical,
lateral, longitudinal and yaw attitude time references $z(t)$,
$y(t)$, $x(t)$ and $\psi(t)$. The latter are supposed to be known
arbitrary time profiles with the only restrictions dictated
by the functional controllability of the system and by the
fulfillment of physical constraints on the control inputs.
Considering equations (1) and the simplified vectored-thrust model in (7) the expressions of the $\phi - \theta$ reference angles
which are compatible with the tracking of the reference signals are given by

$$
\Theta_r := \left( \begin{array}{c}
\phi_r \\
\theta_r
\end{array} \right)
= \left( \begin{array}{c}
\tan \left( \frac{\dot{x}_r}{\dot{y}_r - g} - \frac{C_{\theta} C_{\psi} \dot{y}_r}{\dot{x}_r - g} \right) \\
\tan \left( \frac{\dot{y}_r}{\dot{x}_r - g} + \frac{C_{\phi} \dot{x}_r}{\dot{x}_r - g} \right)
\end{array} \right)
$$

Since the functional controllability of the system requires that
$$
|\theta_r(t)| \leq \frac{\pi}{2} \quad |\phi_r(t)| \leq \frac{\pi}{2} \quad \forall t \geq 0
$$
it turns out that the instantaneous desired accelerations $(\ddot{z}_r(t), \ddot{y}_r(t), \ddot{x}_r(t))$ along the instanton, lateral and longitudinal
direction are restricted according to the instantaneous desired
yaw reference $\psi_d(t)$.

Moreover, by bearing in mind (1), (5) and (8), it is readily
seen that the four desired control inputs $(\tau_e, v_r)$ which
are compatible with the tracking references are given by

$$
\tau_e = I_r \dot{w}_e + K_Q w^2_e \quad c_e = A_2(w^2_e)^{-1} \left( J_z w_z - \tau_e \right)
$$

$$
A_1(w_e, c_e)^{-1} \left[ \begin{array}{c}
Q_{\omega_e} \quad -w_e \\
I_r \quad 0
\end{array} \right] \left( \begin{array}{c}
w_{\omega_e} \\
w_{\phi_e}
\end{array} \right)
$$

with

$$
Q_{\omega_e} = \left( \begin{array}{c}
J_x w_{x_e} - w_z J_y w_{y_e} + w_y J_z w_{x_e} \\
J_y w_{y_e} + w_z J_x w_{x_e} - w_x J_z w_{x_e}
\end{array} \right)
$$
in which the fan angular velocity $w_{\omega_e}$, looking at (7), is given by

$$
w_{\omega_e} = \sqrt{\frac{M}{K_T C_{\phi} C_{\theta}}} - \frac{\ddot{z}_e}{K_T C_{\phi} C_{\theta}}
$$

and with $\omega_f$ the angular velocity of the ducted MAV compatible
with the tracking references given by (see (2))

$$
\omega_f = Q^{-1}(\Theta_r) \left( \begin{array}{c}
\dot{\phi}_r \\
\dot{\theta}_r
\end{array} \right)
$$

We assume that the reference inputs satisfy

$$
\max_{\mu \in \mathcal{I}} ||v_r(t)|| \leq v^U \quad \max_{\mu \in \mathcal{I}} ||\tau_e(t)|| \leq \tau^U
$$

where $v^U$ and $\tau^U$ denote upper bounds on the amplitude of the inputs $v$ and $\tau_e$ imposed by physical constraints. This,
in turn, imposes further constraints which limit the class of
admissible reference signals. Apart these natural limitations
and other minor restrictions specified throughout the paper,
the reference signals are assumed to be completely arbitrary.
We will assume that all state is accessible for control purpose, in particular $w_e$ for engine dynamic, vectors $\Theta_d$ and $w$ for the attitude dynamic, vectors $p$ and $p$ for the vertical,
lateral and longitudinal dynamics. Furthermore the initial
state is supposed to belong to any (arbitrarily large) compact
set with the only restriction that $-\pi/2 < \phi(0) < \pi/2$ and
$-\pi/2 < \theta(0) < \pi/2$ (which implies that the MAV is not
overturned in the initial condition) and $w_e(0) > 0$.

### IV. Control Structure and Main Results

#### A. Vertical and fan dynamics

We concentrate first on the vertical dynamics of the system
given by (see (1) and (7))

$$
M \ddot{z} = -K_T w^2_e \Psi(\Theta) + Mg
$$

where $\Psi(\Theta) := C_{\phi} C_{\theta}$. In the forthcoming discussion
we will argue the existence of a positive constant $\Psi$ such that

$$
\ell(t) := K_T \Psi(\Theta(t)) \geq \Psi \quad \forall t \geq 0
$$

We will show that the existence of this constant is guaranteed
by the design of the control law underlying the attitude
dynamics (see the forthcoming Proposition 1). Defining $x = w^2_e$ it turns out that the overall vertical and fan dynamics can be rewritten as

$$
\begin{align*}
M \ddot{x} &= -\ell(t) x + Mg \\
I_r \dot{x} &= a(t) (-K_Q x + \tau_e)
\end{align*}
$$

in which $a(t) := 2w_e(t)$. The problem of tracking a reference signal $z(t)$ by the state variable $x$ of this system can be reformulated as a state constrained tracking problem for a
chain of integrators. As a matter of fact fix $\bar{x}$ a positive constant such that $\bar{x} < Mg/K_T$ and note that, after
the preliminary control law (assuming, for the time being, $a(t) \geq 2\sqrt{\bar{x}} > 0$)

$$
\tau_e(t) = K_Q x - \frac{1}{\ell(t)a(t)} \left( I_r \dot{x}(t) x(t) + \tau^U_e(t) \right)
$$

For sake of simplicity we do not address robustness issues in the design
of the vertical-fan control law and we assume perfectly known the values of the physical parameters used in (17). The design procedure illustrated in this section can be suitably modified in order to deal with the robust case.

For reason of space we omit details in this direction.
where $\tau'_e(t)$ is a residual control input, system (16) transforms into the following chain of integrators
\begin{equation}
M \ddot{z} = \ddot{x} - I_e \dot{x} = \tau'_e(18)
\end{equation}
in which $\ddot{x}(t) := -\ell(t) x(t) + Mg$, under the state constraint
\begin{equation}
\ddot{x}(t) \leq Mg - K_T \dot{x} > 0.
\end{equation}
As a matter of fact if $\tau'_e$ is designed so that (19) is satisfied for all $t \geq 0$, it turns out that $x(t) \geq \bar{x}$ and $a(t) \geq 2\sqrt{\bar{x}} > 0$ and, as a consequence, (17) is well-defined. Thus the problem of designing a control law for the vertical-fan dynamics can be translated into a problem of tracking a reference signal $\dot{x}$ by the state $\dot{z}$ of (18) fulfilling (19). A number of methods can be used to this purpose. Clearly the fulfillment of (19) requires that the reference signal satisfies
\begin{equation}
M \ddot{z}(t) \leq Mg - K_T \dot{x} - \rho
\end{equation}
for some positive $\rho$ and that the initial condition $\dot{x}(0)$ fulfills the constraint (19). Under these conditions a possible choice of the residual control input $\tau'_e$ can be proven to be
\begin{equation}
\tau'_e(t) = I_e M \ddot{z}^{(3)}(t) - \ddot{x} + \varrho(-\kappa_1 \dot{e}_z - \kappa_2 \dot{e}_z)
\end{equation}
where
\begin{equation}
\varrho(s) = \begin{cases} s & \text{if } s < \rho \\
\rho & \text{otherwise} \end{cases}
\end{equation}
and $\kappa_1, \kappa_2$ are suitable positive constants. Details are omitted for reason of space.

\section*{B. Lateral, longitudinal and attitude dynamics}

\subsection*{1) The control structure and the control law:}
The proposed control structure consists of a cascade control structure constituted by an inner-loop controlling the attitude dynamics and an outer-loop governing the lateral and longitudinal dynamics. The idea of looking at the attitude dynamics as “virtual” control inputs for the lateral and longitudinal dynamics is deeply inspired by [4] and motivated by the particular structure of the lateral and longitudinal dynamics in (1). The goal of the regulator underlying the inner-loop is to control the attitude dynamics in such a way that the ducted MAV does not overturn and the lateral-longitudinal dynamics (having the attitude variables as virtual inputs) asymptotically approach the desired references. This objective is achieved by mixing feedforward control terms, based on the references to be tracked, and high-gain feedback control actions processing the actual attitude measures and the output of the outer lateral-longitudinal controller. The cascade control structure designed is characterized by two-time scale dynamics, with the inner attitude loop as the fast dynamics and the outer lateral/longitudinal loop as the slow dynamics. To impose the two-time scale behavior in the design of outer loop we make use of nested saturation functions providing a decoupling between the attitude and lateral-longitudinal dynamics.

We start with the preliminary choice
\begin{equation}
c = A_{2n}(w_{\dot{c}}^2)^{-1}(\tilde{c} - \tau_e)
\end{equation}
\begin{equation}
\begin{bmatrix} a \\ b \end{bmatrix} = A_{1n}(w_{\dot{c}}^2)^{-1}(\tilde{a} - \dot{\theta})
\end{equation}
in which $\tilde{v} = col(\tilde{\alpha}, \tilde{\beta}, \tilde{c})$ is a residual control input, designed according to the inner-outer loop paradigm discussed above, which is meant to compensate for the nominal part of $A_{n}^{-1}(\cdot)$ and torque $\tau_e$ along body $z$ axis.

\begin{equation}
\begin{aligned}
\dot{\psi}_\psi &= \psi - \psi_t \\
\dot{\Theta}_\psi &= Q(\Theta) \omega \\
\dot{\omega} &= -\text{Skew}(\omega) J_\omega + w_c G_\omega + L(w_{\dot{c}}^2, c) \tilde{\psi} + \\
&+ \Delta(w_{\dot{c}}^2, \tau_e)
\end{aligned}
\end{equation}
\begin{equation}
\Theta_{out} = \lambda_3 \sigma(K_3 \xi_3)
\end{equation}
with
\begin{equation}
\xi_3 := \begin{cases}
\dot{y} - \dot{y}_t \\
\dot{x} - \dot{x}_t \\
\dot{\eta}_y - \dot{\eta}_x
\end{cases} + \lambda_2 \sigma(K_3 \xi_2)
\end{equation}
\begin{equation}
\xi_2 := \begin{cases}
\dot{y} - \dot{y}_t \\
\dot{x} - \dot{x}_t \\
\dot{\eta}_y - \dot{\eta}_x
\end{cases} + \lambda_1 \sigma(K_1 \xi_1)
\end{equation}
\begin{equation}
\xi_1 := \begin{cases}
\dot{\eta}_y - \dot{\eta}_x
\end{cases}
\end{equation}
where $\eta_y$ and $\eta_x$ represent integrator variables governed by
\begin{equation}
\dot{\eta}_y = y - y_t, \quad \dot{\eta}_x = x - x_t.
\end{equation}
In the definition of the outer controller, $(\lambda_i, K_i), i = 1, 2, 3,$ represent design parameters while $\sigma(\cdot)$ is a saturation function defined as any differentiable function $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ satisfying:
\begin{equation}
|\sigma'(s)| := |d\sigma(s)/ds| \leq 2 \text{ for all } s,
\end{equation}
\begin{equation}
\sigma_s(s) > 0 \text{ for all } s \neq 0, \quad \sigma(0) = 0.
\end{equation}
\begin{equation}
\sigma(s) = \text{sgn}(s) \text{ for } |s| \geq 1.
\end{equation}
\begin{equation}
|s| < |\sigma(s)| < 1 \text{ for } |s| < 1.
\end{equation}

\subsection*{2) The inner-loop analysis:}
We start with the study of the inner attitude control loop which, after the preliminary compensation (20) and the addition of the integrator dynamic (22), has the form
\begin{equation}
\begin{aligned}
\dot{\psi}_\psi &= \psi - \psi_t \\
\dot{\Theta}_\psi &= Q(\Theta) \omega \\
J \dot{\omega} &= -\text{Skew}(\omega) J_\omega + w_c G_\omega + L(w_{\dot{c}}^2, c) \tilde{\psi} + \\
&+ \Delta(w_{\dot{c}}^2, \tau_e)
\end{aligned}
\end{equation}
with \( \delta \) as in (21), in which \( L(\cdot) \) and \( \Delta(\cdot) \) are defined as

\[
L(\cdot) = \begin{pmatrix}
L_1(w^2, e) & 0 \\
0 & L_2(w^2)
\end{pmatrix}
\]

\[
\Delta(\cdot) = \begin{pmatrix}
0 \\
0
\end{pmatrix}
\]

with

\[
L_1(\cdot) = I + A_{1\Delta}(\cdot)A_{1o}^{-1}(\cdot) \quad L_2(\cdot) = 1 + A_{2\Delta}(\cdot)A_{2o}^{-1}(\cdot)
\]

**Proposition 1:** Consider the inner attitude loop with \( |\phi(0)| < \pi/2 \) and \( |\theta(0)| < \pi/2 \) and suppose assumption (11) holds true. For any \( K_1 > 0, T^* > 0 \) and \( \varepsilon > 0 \), there exists \( \lambda^*_1 \) such that for all \( K_F \geq \lambda^*_1 \), the following proposition holds:

(i) \( |\phi(t)| < \pi/2 \) and \( |\theta(t)| < \pi/2 \) for all \( t \geq 0 \);

(ii) \( \|([\eta_\Theta(t), \Theta(t), \Theta_r(t), \psi(t) - \psi_r(t)])^T\| \leq \varepsilon \) for all \( t \geq T^* \).

An immediate consequence of Proposition 1 is that \( \Psi(t) > 0 \) for all \( t > 0 \).

3) **The outer-loop analysis:** By looking at system (1), due to results previously discussed about the tuning of the inner loop, the lateral/longitudinal dynamics is described by

\[
M \left( \begin{array}{c} \delta \\ \dot{\delta} \\ \dot{\delta}_T \\
\end{array} \right) = MD(\Theta, \dot{\theta}) \left( \begin{array}{c} T_o \\ 0 \\
\end{array} \right) + n(\Theta, \dot{\theta}) \quad (27)
\]

In which

\[
D(\Theta, \dot{\theta}) = R_{\psi} \left( \begin{array}{c} -1/C_0 \\ 0 \\ 1 \\
\end{array} \right) \left( g - \dot{\zeta}_r \right)
\]

\[
n(\cdot) = \left( \begin{array}{c} 1 \\ 0 \\ 0 \\
\end{array} \right) R_\psi \left( \begin{array}{c} T_o/C_0 \\ 0 \\
\end{array} \right) \xi(t) \quad (28)
\]

in which \( R_\psi \) is a two dimensional rotation matrix around \( \psi \). The term \( n(\cdot) \) represent a vanishing coupling term with vertical and fan dynamics. For the overall closed-loop system the following proposition holds:

**Proposition 2:** Consider system (27), (26), (20)-(25). Let \( K_i^1, K_i^2, K_i^3 \) be such that the following inequalities are satisfied

\[
\frac{\lambda^*_i}{K_i^2} < \frac{\lambda^*_i}{4}, \quad \frac{\lambda^*_i}{K_i^3} < \frac{\lambda^*_i}{4}, \quad 4K_i^1\lambda^*_i < \frac{\lambda^*_i}{4}, \quad 4K_i^2\lambda^*_i < \frac{\lambda^*_i}{8}
\]

and

\[
24K_i^1 < \frac{1}{6}, \quad 24K_i^2 < \frac{1}{6} \mu^2_i, \quad 24K_i^3 < \frac{1}{6} \mu^2_i \quad (29)
\]

With

\[
\mu^2_i := M^L(g - \|\zeta_\Theta(\cdot)\|) > 0
\]

\[
\mu^2_i := M^U(g - \|\zeta_\theta(\cdot)\|) > 0
\]

Moreover, let \((K_i, \lambda_i)\) be chosen as

\[
\lambda_i = \frac{1}{4} \lambda_i \quad \text{and} \quad K_i = \epsilon K_i^*, \quad i = 1, 2, 3 \quad (31)
\]

in which \( \varepsilon \) is a positive design parameters. Let \( R_\Delta \) be an arbitrary positive number. There exist \( r_1 > 0, r_2 > 0, R_n, \) \( \epsilon^* > 0 \) and \( K_D^2 > 0 \) such that, for any positive \( K_D \leq K_D^2 \) and \( \epsilon \leq \epsilon^* \), there exists \( K_P^2(\epsilon, K_D) \) such that for any \( K_D \geq K_P^2(\epsilon, K_D) \) the following proposition holds:

\( |n(\cdot), \Delta(\cdot)| \) and linear asymptotic gains (see [4] and [7]); in particular if \( \|n\|_4 < R_n\epsilon^2 \) and \( \|\Delta\| \leq R_\Delta \) then the overall state satisfies the following asymptotic bound

\[
\|([\xi_1, \xi_2, \xi_3, \eta_\Theta, \Theta - \Theta_r, w - w_r])\|_a \leq \frac{\Delta_\epsilon}{R_\Delta} \|\xi\|_a \quad (32)
\]

In summary, in case of perfect knowledge of the ducted MAV dynamics, it turns out that \( \Delta_\epsilon(\cdot) \equiv 0 \) by which it is possible to conclude that perfect asymptotic tracking of the references \((y_r, x_r, z_r, \psi_r)\) is achieved. On the other hand, in presence of uncertainties, a residual tracking error is observed which, though, can be rendered arbitrarily small by increasing the parameter \( K_P \).

**V. CONCLUSION**

We presented a nonlinear robust controller for a ducted-fan MAV in order to track some vertical, lateral, longitudinal and yaw attitude references with only some restrictions on the higher order time behavior. The ducted MAV proposed consists of a main rotor and four active aerodynamic surfaces controlled to compensate the engine torque and to direct the propeller main thrust in the desired direction.

**REFERENCES**


