Distributed Port Hamiltonian Formulation of the Timoshenko Beam: Modeling and Control

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Abstract. The purpose of this paper is to show how the Timoshenko beam can be fruitfully treated within the framework of distributed port Hamiltonian systems (dpH systems), both for modeling and control purposes. In this manner, rather simple and elegant considerations can be drawn regarding both the modeling and control of this mechanical system. In particular, it is shown how control approaches already presented in the literature can be elegantly unified, and a new control methodology is presented and discussed.

1 Introduction

Flexible beams are generally modeled according to the classical Euler-Bernoulli theory: this formulation provides a good description of the dynamical behavior of the system if the beam’s cross sectional dimension are small in comparison of its length. In this case, the effects of the rotary inertia of the beam are not considered. A more accurate beam model is provided by the Timoshenko theory, according to which the rotary inertia and also the deformation due to shear are considered. The resulting Timoshenko model of the beam is generally more accurate in predicting the beam’s response than the Euler-Bernoulli one, but, on the other hand, it is more difficult to utilize for control purposes because of its complexity.

Clearly, the dpH [1] formulation of the Timoshenko model of the beam (see also [2]) does not reduce the complexity of the model itself, but it is useful both for modeling considerations and control purposes. From the modeling point of view, the internal and external interconnections of the system are revealed: it is clear how the kinetic and potential elastic energy domains interact and how the system can exchange power with the environment through its border and/or a distributed port. Moreover, the dpH representation of the system makes it possible to extend well-established passive control strategies that were originally developed for finite dimensional port Hamiltonian systems.

This paper is organized as follows: in Sec. 2 the dpH model of the Timoshenko beam is presented and the underlying Dirac structure is shown. Then, in Sec. 3 and Sec. 4 some considerations about control strategies of the beam are presented. In particular, in Sec. 3 the well-known control by damping injection [4] is extended to distributed parameter systems in order to stabilize the beam acting through its boundary; the results presented in [5, 6] are now obtained in this new framework of dpH systems. In Sec. 4 the energy shaping by interconnection control technique [1] [7, 8] is extended to distributed parameter systems in order to control a mechanical system made of a flexible beam with a mass connected at one of its extremity. The finite dimensional controller, acting on the system through the other extremity, is developed by properly extending the concept of Casimir functions to the infinite dimensional case, as already done in [9] for a transmission line. Finally, conclusions and suggestions for future work are illustrated in Sec. 5.

2 dpH description of the Timoshenko beam

2.1 Background. The classical formulation

According to the Timoshenko theory, the motion of a beam can be described by the following system of PDE:

$$\rho \frac{\partial^2 w}{\partial t^2} - K \frac{\partial^2 w}{\partial x^2} + K \frac{\partial \phi}{\partial x} = 0$$

$$I_\rho \frac{\partial^2 \phi}{\partial t^2} - EI \frac{\partial^2 \phi}{\partial x^2} + K \left( \phi - \frac{\partial w}{\partial x} \right) = 0$$  \hspace{1cm} (1)

where $t$ is the time and $x \in [0, L]$ is the spatial coordinate along the beam in its equilibrium position, $w(x, t)$ is the deflection of the beam from the equilibrium configuration and $\phi(x, t)$ is the rotation of the beam’s cross section due to bending; the motion takes place in the $wx$-plane.

The coefficients $\rho$, $I_\rho$, $E$ and $I$, assumed to be constant, are the mass per unit length, the mass moment of inertia of the cross section, Young’s modulus and the moment of inertia of the cross section, respectively.
The coefficient $K$ is equal to $kGA$, where $G$ is the modulus of elasticity in shear, $A$ is the cross sectional area and $k$ is a constant depending on the shape of the cross section.

The mechanical energy is given by the following relation

$$
\mathcal{H}(t) := \frac{1}{2} \int_0^L \rho \left( \frac{\partial w}{\partial t} \right)^2 + I_p \left( \frac{\partial \phi}{\partial t} \right)^2 \, dx + \frac{1}{2} \int_0^L K \left( \phi - \frac{\partial w}{\partial x} \right)^2 + EI \left( \frac{\partial \phi}{\partial x} \right)^2 \, dx
$$

(2)

that points out that, as every infinite dimensional system, the beam is characterized by a spatial domain $\mathbb{D} := [0, L]$, with border $\partial \mathbb{D} = \{0, L\}$, and by the presence of two interactive energy domains, the kinetic and the potential elastic.

### 2.2 Timoshenko beam in dpH form

The starting point in the definition of a port Hamiltonian system (both finite and infinite dimensional) is the identification of a suitable space of power (or energy) variables, strictly related to the geometry of the system, and the definition of a Dirac structure on this space of power variables, in order to describe the internal and external interconnection of the system.

The potential elastic energy in (2) is a function of the shear and of the bending, given by the following 1-forms:

$$
\epsilon_t(t, x) = \epsilon_t(t, x) \, dx = \left[ \frac{\partial w}{\partial x} (t, x) - \phi(t, x) \right] \, dx \quad \epsilon_r(t, x) = \epsilon_r(t, x) \, dx = \frac{\partial \phi}{\partial x} (t, x) \, dx
$$

(3)

The associated co-energy variables are the 0-forms (functions) shear force and the bending momentum, given by $\sigma_t(t, x) = K \epsilon_t(t, x) = K \left[ \frac{\partial w}{\partial x} (t, x) - \phi(t, x) \right]$ and $\sigma_r(t, x) = EI \epsilon_r(t, x) = EI \frac{\partial \phi}{\partial x} (t, x)$, where $*$ is the Hodge star operator defined, for example, in [19]. Besides, the kinetic energy is the following (translational and rotational) momenta, i.e. of the following 1-form:

$$
p_t(t, x) = p_t(t, x) \, dx = \rho \frac{\partial w}{\partial t} (t, x) \, dx \quad p_r(t, x) = p_r(t, x) \, dx = I_p \frac{\partial \phi}{\partial t} (t, x) \, dx
$$

(4)

and the associated co-energy variables are the 0-forms translational and rotational momenta, given by $\psi_t(t, x) = \frac{1}{\rho} * p_t(t, x) = \frac{\partial w}{\partial t} (t, x)$ and $\psi_r(t, x) = \frac{1}{I_p} * p_r(t, x) = \frac{\partial \phi}{\partial t} (t, x)$.

If $\mathcal{N}$ is an $n$-dimensional (Riemannian) manifold, the space of $k$-forms on $\mathcal{N}$, i.e. the space of $k$-linear alternating functions, is given by $\Omega^k(\mathcal{N})$. So, we have that $p_t, p_r, \epsilon_t, \epsilon_r \in \Omega^1(\mathbb{D})$ and that $w, \phi \in \Omega^2(\mathbb{D})$.

If $d$ indicates the exterior derivative on the space of forms, it is possible to re-write (3) and (4) as

$$
p_t = \rho \frac{\partial w}{\partial t} \quad \epsilon_t = \frac{\partial w}{\partial x} - \phi \quad p_r = I_p \frac{\partial \phi}{\partial t} \quad \epsilon_r = \frac{\partial \phi}{\partial x}
$$

and the total energy (2) becomes the following (quadratic) functional:

$$
\mathcal{H}(p_t, p_r, \epsilon_t, \epsilon_r) = \int_{\mathbb{D}} H(p_t, p_r, \epsilon_t, \epsilon_r) = \frac{1}{2} \int_{\mathbb{D}} \left( \frac{1}{\rho} * p_t \wedge \delta_p \frac{\partial p_t}{\partial t} + \frac{1}{I_p} * p_r \wedge \delta_p \frac{\partial p_r}{\partial t} + K \epsilon_t \wedge \epsilon_t + EI \epsilon_r \wedge \epsilon_r \right)
$$

(5)

with $H : \Omega^1(\mathbb{D}) \times \Omega^1(\mathbb{D}) \times \Omega^1(\mathbb{D}) \times \Omega^1(\mathbb{D}) \to \Omega^1(\mathbb{D})$ the energy density. Consider a time function $(p_t(t), p_r(t), \epsilon_t(t), \epsilon_r(t)) \in \Omega^1(\mathbb{D}) \times \Omega^1(\mathbb{D}) \times \Omega^1(\mathbb{D}) \times \Omega^1(\mathbb{D})$, with $t \in \mathbb{R}$, and evaluate the energy $H(t, \cdot, \cdot, \cdot)$ along this trajectory. At any time $t$, the variation of internal energy, that is the power exchanged with the environment, is given by

$$
\frac{d\mathcal{H}}{dt} = \int_{\mathbb{D}} \left( \delta_p \left( \frac{\partial p_t}{\partial t} \right) \wedge \delta_p \mathcal{H} + \delta_p \mathcal{H} \wedge \frac{\partial p_t}{\partial t} + \delta_p \left( \frac{\partial \epsilon_t}{\partial t} \right) \wedge \delta_p \mathcal{H} + \delta_p \mathcal{H} \wedge \frac{\partial \epsilon_t}{\partial t} \right) 
$$

$$
= \int_{\mathbb{D}} \left( \frac{1}{\rho} * p_t \wedge \frac{\partial p_t}{\partial t} + \frac{1}{I_p} * p_r \wedge \frac{\partial p_r}{\partial t} + (K \epsilon_t) \wedge \frac{\partial \epsilon_t}{\partial t} + (EI \epsilon_r) \wedge \frac{\partial \epsilon_r}{\partial t} \right)
$$

(6)

The differential forms $\frac{\partial p_t}{\partial t}, \frac{\partial p_r}{\partial t}, \frac{\partial \epsilon_t}{\partial t}$ and $\frac{\partial \epsilon_r}{\partial t}$ are the time derivatives of the energy variables $p_t, p_r, \epsilon_t, \epsilon_r$ and represent the generalized velocities (flows), while $\delta_p \mathcal{H}, \delta_p \mathcal{H}, \delta_p \mathcal{H}, \delta_p \mathcal{H}$ are the variational derivative of the total energy $H : \Omega^1(\mathbb{D}) \times \Omega^1(\mathbb{D}) \times \Omega^1(\mathbb{D}) \times \Omega^1(\mathbb{D}) \to \mathbb{R}$. They are related to the rate of change of the stored energy and represent the generalized forces (efforts).
The dpH formulation of the Timoshenko beam can be obtained either by expressing (1) in terms of \( p_t, p_r, \varepsilon_r \) and \( \varepsilon_t \) introduced in (3) and (4), or, in a more rigorous way, by revealing the underlying Dirac structure of the model. For this purpose, it is necessary to define the space of power variables. The space of flows is given by

\[
\mathcal{F} := \underbrace{\Omega^1(\mathcal{D}) \times \Omega^1(\mathcal{D}) \times \Omega^1(\mathcal{D}) \times \Omega^0(\partial \mathcal{D}) \times \Omega^0(\partial \mathcal{D})}_{\text{generalized velocities}} \times \underbrace{\Omega^0(\partial \mathcal{D})}_{\text{border flow}}
\]

and it is well known that the space of effort \( \mathcal{E} \) is the dual of \( \mathcal{F} \). The concept of duality over the space of forms can be given by the following:

**Proposition 2.1.** Suppose that \( \mathcal{N} \) is an \( n \)-dimensional manifold. Then, the dual space \( (\Omega^k(\mathcal{N}))^* \) of \( \Omega^k(\mathcal{N}) \) can be identified with \( \Omega^{n-k}(\mathcal{N}) \) and the duality product between \( \Omega^k(\mathcal{N}) \) and \( (\Omega^k(\mathcal{N}))^* \) by

\[
\langle \beta, \alpha \rangle := \int_{\mathcal{N}} \beta \wedge \alpha
\]

with \( \alpha \in \Omega^k(\mathcal{N}) \) and \( \beta \in \Omega^{n-k}(\mathcal{N}) \). The same result holds for \( \Omega^k(\partial \mathcal{N}) \).

**Proof.** The proof can be found in [1].

An immediate consequence of Prop. 2.1 is that the dual space \( \mathcal{E} \) of \( \mathcal{F} \), the space of efforts, can be easily identified with

\[
\mathcal{E} := \underbrace{\Omega^0(\partial \mathcal{D}) \times \Omega^0(\partial \mathcal{D}) \times \Omega^0(\partial \mathcal{D}) \times \Omega^3(\partial \mathcal{D}) \times \Omega^3(\partial \mathcal{D})}_{\text{generalized forces}} \times \underbrace{\Omega^0(\partial \mathcal{D})}_{\text{border effort}}
\]

Given the duality product defined in (5), it is possible to introduce the +pairing operator on the space of power variables by means of the following definition that can be easily specialized for the Timoshenko beam case.

**Definition 2.1 (+pairing operator).** Suppose that \( (\alpha_i, \beta_i) \in \Omega^k(\mathcal{N}) \times \Omega^{n-k}(\mathcal{N}), i = 1, 2 \). Then

\[
\ll (\alpha_1, \beta_1), (\alpha_2, \beta_2) \gg := \langle \beta_2, \alpha_1 \rangle + \langle \beta_1, \alpha_2 \rangle = \int_{\mathcal{N}} (\beta_2 \wedge \alpha_1 + \beta_1 \wedge \alpha_2)
\]

Since the space of power variables and the +pairing operator are specified, it is possible to give the following:

**Definition 2.2 (Dirac structure).** Suppose that \( \mathcal{F} \) and \( \mathcal{E} \) are linear spaces with a dual pairing \( \langle \cdot, \cdot \rangle \).

A Dirac structure is a linear subspace \( \mathbb{D} \subset \mathcal{F} \times \mathcal{E} \) such that \( \mathbb{D} = \mathbb{D}^\perp \), with \( \perp \) denoting the orthogonal complement with respect to the +pairing \( \ll \cdot, \cdot \gg \).

**Note 2.1.** The previous definition is quite general and can be reformulated as follows. Given \( (f_i, e_i) \in \mathbb{D} \subset \mathcal{F} \times \mathcal{E}, i = 1, 2 \), then \( \ll (f_1, e_1), (f_2, e_2) \gg = 0 \). It is immediate that, if \( (f, e) \in \mathbb{D} \), then

\[
0 = \ll (f, e), (f, e) \gg = \langle f, f \rangle + \langle e, e \rangle \quad \text{(10)}
\]

In other words, if \( (f, e) \in \mathcal{F} \times \mathcal{E} \) is a couple of power conjugated variables, the fact that they belong to the Dirac structure \( \mathbb{D} \) implies power conservation, i.e. the dual product is equal to 0. The Dirac structure is the geometrical tool by means of which it is possible to deal with power conserving interconnection in physical systems. Once a proper interconnection structure is defined, the port Hamiltonian model of a physical system, both finite and infinite dimensional, follows automatically.

With the following proposition, the main result of this section is presented.

**Proposition 2.2 (the Timoshenko beam Dirac structure).** Consider the space of power variables \( \mathcal{F} \times \mathcal{E} \) with \( \mathcal{F} \) and \( \mathcal{E} \) defined in (3) and (4) and the bilinear form (+pairing operator) \( \ll \cdot, \cdot \gg \) given by (5). Define the following linear subspace \( \mathbb{D} \) of \( \mathcal{F} \times \mathcal{E} \):

\[
\mathbb{D} = \{(f_p, f_p, f_t, f_e, f_b, e_p, e_p, e_t, e_t, e_e, e_b, e_p) \in \mathcal{F} \times \mathcal{E} \mid \begin{bmatrix} f_p & 0 & 0 & 0 & e_p \\ f_p & 0 & 0 & d & e_p \\ f_t & d & \ast & 0 & 0 & e_t \\ f_e & 0 & d & 0 & e_e \end{bmatrix}, \begin{bmatrix} f_b^t \\ f_b^e \\ e_b^t \\ e_b^e \end{bmatrix} = \begin{bmatrix} e_p |_{\partial \mathcal{D}} \\ e_p |_{\partial \mathcal{D}} \\ e_t |_{\partial \mathcal{D}} \\ e_e |_{\partial \mathcal{D}} \end{bmatrix} \}
\]

where \( |_{\partial \mathcal{D}} \) denotes the restriction on the border of the (spatial) domain \( \mathcal{D} \). Then \( \mathbb{D} = \mathbb{D}^\perp \), that is \( \mathbb{D} \) is a Dirac structure.
the controller can interact with the system through the border, energy dissipation can be introduced supplied through the border.

In this section, some considerations about control by damping injection \cite{4} applied to the Timoshenko beam are presented. The energy functional \eqref{5} assumes its minimum in the zero configuration, i.e. when \( p_t = 0, p_r = 0, \epsilon_t = 0 \) and \( \epsilon_r = 0 \) or, equivalently, when

\[
 w(t, x) = \alpha^* x + d^* \quad \phi(t, x) = \alpha^*
\]

where the constants \( \alpha^* \) and \( d^* \) are determined by the boundary conditions on \( w(\cdot, \cdot) \) and \( \phi(\cdot, \cdot) \): \( \alpha^* \) represents the rotation angle of the beam around the point \( x = 0 \) and \( d^* \) is the vertical displacement in \( x = 0 \). If a dissipation effect is introduced by means of a controller, it is possible to drive the state of the beam to the configuration where the (open loop) energy functional \eqref{5} assumes its minimum. If the controller can interact with the system through the border, energy dissipation can be introduced
These results were already presented in [5] using a different approach. The proposed control law was written in the following form:

$$\begin{align*}
\begin{cases}
\dot{f}_b(t, L) &= -b'(t) \cdot e_b(t, L) \\
\dot{f}_r(t, L) &= -b'(t) \cdot e_r(t, L)
\end{cases}
\end{align*}$$

(17)

with \(b'(\cdot), b''(\cdot) > 0\) smooth functions.

In this way, the energy balance equation (15) becomes

$$\frac{dH_c}{dt}(t) = -b'(t) [K \cdot \epsilon_t \mid_{x=L}]^2 - b''(t) [EI \cdot \epsilon_r \mid_{x=L}]^2 + [e_b(t, 0) f_b(t, 0) + e_r(t, 0) f_r(t, 0)]$$

(18)

If, for example, the boundary conditions in \(x = 0\) are \(w(t, 0) = 0\) and \(\phi(t, 0) = 0\), then \(f_b(t, 0) = f_b(t, 0) = 0\) and (15) becomes

$$\frac{dH_c}{dt}(t) = -b'(t) [K \cdot \epsilon_t \mid_{x=L}]^2 - b''(t) [EI \cdot \epsilon_r \mid_{x=L}]^2 \leq 0$$

Since in (16) necessarily \(\alpha^* = 0\) and \(d^* = 0\) in order to be compatible with the boundary conditions in \(x = 0\), we deduce that the total energy [3] reaches its minimum and the beam assumes the configuration \(w(t, x) = 0\) and \(\phi(t, x) = 0\).

These results were already presented in [5] using a different approach. The proposed control law was written in the following form:

$$\begin{align*}
\frac{\partial w}{\partial t}(t, L) &= -b'(t) \cdot K \left[ \frac{\partial w}{\partial x}(t, L) - \phi(t, L) \right] \\
\frac{\partial \phi}{\partial t}(t, L) &= -b''(t) \cdot EI \frac{\partial \phi}{\partial x}(t, L)
\end{align*}$$

which is clearly equivalent to (17). The main advantage in approaching the problem within the framework of dpH systems is that both the design of the control law and the proof of its stability can be presented in a more intuitive (in some sense physical) and elegant way.

4 Energy shaping through the boundary

Consider the mechanical system of Fig. 1(a) in which a flexible beam is connected in \(x = L\) to a rigid body with mass \(m\) and inertia momentum \(J\) and to a controller in \(x = 0\). The controller acts on the system with a force \(f_c\) and a torque \(\tau_c\). Since the Timoshenko model of the beam is valid only for small deformations, it is possible to assume that the motion of the mass is the combination of both rotation and translation along \(x = L\). The port Hamiltonian model of the mass is given by:

$$\begin{align*}
\begin{bmatrix}
\dot{q} \\
\dot{p}
\end{bmatrix} &= \left( \begin{bmatrix}
0 & I \\
- I & 0
\end{bmatrix} - \begin{bmatrix}
0 & 0 \\
0 & D
\end{bmatrix} \right) \left[ \frac{\partial q}{\partial p} \frac{H_c}{p} \right] + \begin{bmatrix}
0 \\
1
\end{bmatrix} f_c \\
\begin{bmatrix}
q_c \\
\dot{p}_c
\end{bmatrix} &= \left( \begin{bmatrix}
0 & I \\
- I & 0
\end{bmatrix} - \begin{bmatrix}
0 & 0 \\
D_c & 0
\end{bmatrix} \right) \left[ \frac{\partial q_c}{\partial p_c} H_c \right] + \begin{bmatrix}
0 \\
G_c
\end{bmatrix} f_c
\end{align*}$$

(19)

with \(D = D^T \geq 0\), where \(q = [q_1, q_2]^T \in Q\) are the generalized coordinates, with \(q_1\) the distance from the equilibrium configuration and \(q_2\) the rotation angle, \(p \in T^*Q\) are the generalized momenta and \(H(p, q) = \frac{1}{2} p \cdot M^{-1} \cdot p + \frac{1}{2} \left( \frac{q_1^2}{m} + \frac{q_2^2}{J} \right)\) is the total energy (Hamiltonian). Moreover, the port variables are given by \(f_c, e_c \in \mathbb{R}^2\).

As regard the controller, we assume that it can be modeled by means of the following finite dimensional port Hamiltonian systems

$$\begin{align*}
\begin{bmatrix}
\dot{q}_c \\
\dot{p}_c
\end{bmatrix} &= \left( \begin{bmatrix}
0 & I \\
- I & 0
\end{bmatrix} - \begin{bmatrix}
0 & 0 \\
D_c & 0
\end{bmatrix} \right) \left[ \frac{\partial q_c}{\partial p_c} H_c \right] + \begin{bmatrix}
0 \\
G_c
\end{bmatrix} f_c \\
e_c &= G_c^T \frac{\partial p_c}{\partial p_c} H_c
\end{align*}$$

(20)
with \( D_c = D_c^T \geq 0 \), where \( q_c \in Q_c \) are the generalized coordinates, with \( \dim(Q_c) = 2 \), \( p_c \in \mathbb{R}^2 \) are the power conjugated port variables. Moreover, \( H_c(q_c, p_c) \) is the Hamiltonian and it will be specified in the remaining part of this section in order to drive the whole system in a desired equilibrium configuration.

The port causality of both the mass and the controller is assumed to be with flows as input and efforts as outputs. As pointed out in [12], it is possible to interconnect two port Hamiltonian systems only if a port dualization is applied on one of the system. In this way, a system can have an effort as input and a flow as output. The bond graph representation of the closed loop system made of the Timoshenko beam, the mass in \( x = L \) and the finite dimensional port Hamiltonian controller acting in \( x = 0 \) is given in Fig. 2, where \text{SGY} \ is the symplectic gyrator that implements the port dualization. Then, the interconnections

\[
\begin{align*}
H_c & \quad \text{SGY} \quad H_c \quad \text{SGY} \quad 1 \quad H
\end{align*}
\]

Figure 2: Bond graph representation of the closed-loop system.

constraints between the port variables of the subsystems are given by the following power-preserving relations:

\[
\begin{bmatrix}
q^\prime_k(L) \\
p^\prime_k(L)
\end{bmatrix}^T = -e \\
\begin{bmatrix}
q^\prime_k(0) \\
p^\prime_k(0)
\end{bmatrix}^T = e_c \quad \begin{bmatrix}
f^\prime_k(L) \\
e^\prime_k(L)
\end{bmatrix}^T = f \\
\begin{bmatrix}
f^\prime_k(0) \\
e^\prime_k(0)
\end{bmatrix}^T = f_c
\]

From (14), (19), (20) and (21), it is possible to obtain the mixed finite and infinite dimensional port Hamiltonian representation of the closed-loop system. The total energy \( H_{cl} \) is defined in the extended space \( \chi := [q, p, q_c, p_c, p_t, p_r, \epsilon_r, \epsilon_t]^T \) and it is given by the sum of the energy functions of the subsystems, that is

\[
H_{cl} := H + H_c + H
\]

Moreover, it is easy to verify that the energy rate is equal to

\[
\frac{dH_{cl}}{dt} = - \left( \frac{\partial^T H_{cl}}{\partial p} \frac{\partial H_{cl}}{\partial p_c} + \frac{\partial^T H_{cl}}{\partial p_c} \frac{\partial H_{cl}}{\partial p_c} \right)
\]

where \( D_c \) and \( H_c \) have to be designed in order to drive the system in the desired equilibrium position, in general different from the trivial one, for which only some damping injection, i.e. \( D_c > 0 \), is enough. The basic idea is to shape the total energy \( H_{cl} \) by properly choosing the controller Hamiltonian \( H_c \) in order to have a new minimum of energy in the desired configuration that can be reached if some dissipative effect is introduced.

Clearly, the controller Hamiltonian can be freely assigned, but, in general, the energy variables of the controller are not related to the energy variables of the plant and, as a consequence, it is not clear how the total energy can be shaped and the desired minimum assigned. The first step is to relate the energy variables of the controller to the energy variables of the plant independently from the Hamiltonian of the controller and, if it is possible, from the Hamiltonian of the plant. These invariants are called, in the
finite dimensional framework, Casimir functions, [7, 8]. The Casimir functions are structural invariants in the sense that they are invariant independently from the Hamiltonian of the system. In the case of mixed finite and infinite dimensional system, it is possible to give the following:

**Definition 4.1 (Casimir functionals).** Consider a mixed finite and infinite dimensional port Hamiltonian system with state space given by \( X \times X_\infty \), where \( X \) is the state space of the finite dimensional part and \( X_\infty \) the state space of the infinite dimensional one, and Hamiltonian \( H : X \times X_\infty \to \mathbb{R} \). A functional \( C : X \times X_\infty \to \mathbb{R} \) is a Casimir functional for the system if and only if

\[
\frac{dC}{dt} = 0, \quad \forall H : X \times X_\infty \to \mathbb{R} \text{ Hamiltonian of the system}
\]

that is, \( C \) is constant independently of the Hamiltonian of the system.

**Note 4.1.** For the system of Fig. 1, we have that \( X = Q \times T^* Q \times Q_\infty \) and \( X_\infty = \Omega^1(D) \times \Omega^1(D) \times \Omega^1(D) \), with Hamiltonian given by (22).

Given \( \chi := [q, p, q_c, p_c, p_t, p_re, \epsilon_t] \) \( T \in X \times X_\infty \) and a functional \( C : X \times X_\infty \to \mathbb{R} \), we have that

\[
\frac{dC}{dt} = \frac{\partial^T C}{\partial q} \frac{dq}{dt} + \frac{\partial^T C}{\partial p} \frac{dp}{dt} + \frac{\partial^T C}{\partial q_c} \frac{dq_c}{dt} + \frac{\partial^T C}{\partial p_c} \frac{dp_c}{dt} + \int_D \left( \delta_{p_c} C + \frac{\partial p_c}{\partial t} \right) + \int_D \left( \delta_{p_t} C + \frac{\partial p_t}{\partial t} \right) + \delta_{\epsilon_t} C + \frac{\partial \epsilon_t}{\partial t}
\]

that has to be equal to zero for every Hamiltonian \( H, H_t \) and \( H_t \). It is possible to prove that this is true if and only if

\[
\begin{align*}
\frac{d\delta_{q} C}{dt} &= 0, \\
\frac{d\delta_{q_c} C}{dt} &= 0, \\
\frac{d\delta_{p} C}{dt} &= 0, \\
\frac{d\delta_{p_c} C}{dt} &= 0
\end{align*}
\]

In other words, the following proposition holds.

**Proposition 4.1.** Consider the mixed finite and infinite dimensional port Hamiltonian system of Fig. 2, that is the result of the power conserving interconnection (21) of the subsystems (14), (19) and (20). If \( X \times X_\infty \) is the extended state space of the system, as introduced in Note 4.1, then a functional \( C : X \times X_\infty \to \mathbb{R} \) is a Casimir for the closed-loop system if and only if relations (22) hold.

In order to control the flexible beam with the finite dimensional controller (20), the first step is to find Casimir functionals for the closed-loop system that can relate the state variables of the controller \( q \) to the state variables that describe the configuration of the flexible beam and of the mass connected to its extremity. In particular, we are looking for some functionals \( \tilde{C}_i \), \( i = 1, 2 \), such that \( C_i(q, p, q_c, p_c, p_t, p_re, \epsilon_t, \epsilon_r) := q_{c_i} - \tilde{C}_i(q, p, q_c, p_c, p_t, p_re, \epsilon_t, \epsilon_r) \), with \( i = 1, 2 \), are Casimir functionals for the closed loop system, that is they satisfies the conditions of Prop. 4.1.

First of all, from (22), it is immediate to note that every Casimir functional cannot depend on \( p \) and \( p_c \). Moreover, since it is necessary that \( d\delta_{q_i} C_i = 0 \) and \( d\delta_{p_i} C_i = 0 \), we deduce that \( \delta_{p_i} C_i \) and \( \delta_{p_c} C_i \) have to be constant as function on \( x \) on \( D \) and their value will be determined by the boundary conditions on \( C_i \). Since, from (22), \( \delta_{p_c} C_i \mid_{D} = 0 \), we deduce that \( \delta_{p_i} C_i = 0 \) on \( D \). But \( d\delta_{p_i} C_i = \ast \delta_{p_i} C_i = 0 \), then, from the boundary conditions, we deduce that also \( \delta_{p_i} C_i = 0 \) on \( D \). As a consequence, all the admissible Casimir functionals are also independent from \( p_t \) and \( p_re \), that is \( C_i(q, q_c, \epsilon_t, \epsilon_r) := q_{c_i} - \tilde{C}_i(q, \epsilon_t, \epsilon_r) \), with \( i = 1, 2 \).

Assuming \( G_c = I \), we have that

\[
\frac{\partial C_1}{\partial q_c} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \delta_{q_c} C_1 \mid_{x=0} \quad \frac{\partial C_1}{\partial \epsilon_r} = \delta_{q_r} C_1 \mid_{x=0}
\]

and, consequently, \( \delta_{q_c} C_1 = 1 \) on \( D \). From (22), we have that \( d\delta_{q} C_1 = -\delta_{q} C_1 = -\ast 1 = -dx \); then, \( \delta_{q} C_1 = -x+c_1 \), where \( c_1 \) is determined by the boundary conditions. Since, from (21), \( \delta_{q} C_1 \mid_{x=0} = 0 \), then \( c_1 = 0 \); moreover, we deduce that \( \delta_{q} C_1 \mid_{x=L} = -L \), relation that introduces a new boundary condition in \( x = L \). A consequence is that

\[
\frac{\partial C_1}{\partial q} = \begin{bmatrix} \delta_{q} C_1 \mid_{x=L} \\ \delta_{q} C_1 \mid_{x=L} \end{bmatrix} = \begin{bmatrix} 1 \\ -L \end{bmatrix}
\]
The new equilibrium configuration for the beam is given by

\[ \mathcal{C}_1(q, q_c, \epsilon_t, \epsilon_r) = q_{c,1} - (Lq_2 - q_1) - \int_D (x\epsilon_r - \epsilon_t) \] (25)

a Casimir for the closed loop system. Following the same procedure, it is possible to calculate \( \mathcal{C}_2 \). From \cite{23}, we have that

\[ \frac{\partial \mathcal{C}_2}{\partial q_c} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \delta_c \mathcal{C}_2 \big|_{x=0} \\ \delta_c \mathcal{C}_2 \big|_{x=0} \end{bmatrix} \] (26)

and then \( \delta_c \mathcal{C}_2 = 0 \) on \( D \); moreover, \( d\delta_c \mathcal{C}_2 = 0 \) and, consequently, \( \delta_c \mathcal{C}_2 = 1 \) on \( D \) since (26) holds. Again from \cite{23}, we deduce that

\[ \frac{\partial \mathcal{C}_2}{\partial q} = \begin{bmatrix} \delta_c \mathcal{C}_2 \big|_{x=L} \\ \delta_c \mathcal{C}_2 \big|_{x=L} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]

So we can state that

\[ \mathcal{C}_2(q, q_c, \epsilon_t, \epsilon_r) = q_{c,2} + q_2 + \int_D \epsilon_r \] (27)

is another Casimir functionals for the closed loop system. In conclusion, the following proposition has been proved.

**Proposition 4.2.** Consider the mixed finite and infinite dimensional port Hamiltonian system of Fig. 2 that is the result of the power conserving interconnection \cite{21} of the subsystems \cite{14}, \cite{12} and \cite{20}. Then \cite{23} and \cite{27} are Casimir functionals for this system.

**Note 4.2.** Since \( \mathcal{C}_i, i = 1, 2 \), are Casimir functionals, they are invariant for the system of Fig. 2. Then, for every energy function \( H_c \) of the controller, we have that

\[ q_{c,1} = (Lq_2 - q_1) + \int_D (x\epsilon_r - \epsilon_t) + C_1 \]

\[ q_{c,2} = -q_2 - \int_D \epsilon_r + C_2 \]

where \( C_1 \) and \( C_2 \) depend on the initial conditions. If the initial configuration of the system is known, then it is possible to assume these constants equal to zero. Since \( H_c \) is an arbitrary function of \( q_c \), it is possible to shape the total energy function of the closed-loop system in order to have a minimum of energy in a desired configuration: if some dissipation effect is present, the new equilibrium configuration will be reached.

The desired equilibrium configuration is represented in Fig. 1(b). The extremity of the beam in \( x = 0 \) can translate in \( q^* \) and the beam can rotate of an angle equal to \( \alpha^* \). Note that the Timoshenko model of the beam remains valid if the rotational angle due to bending remains small. Then, in practical applications, it is necessary to assume the bending deformations close to zero. On the other side, it is important to point out that, from a mathematical point of view, the proposed controller will assure stability for every \( \alpha^* \).

The new equilibrium configuration for the beam is given by

\[ w(t, x) = w^*(t, x) = q^* + x\alpha^* \]

\[ \phi(t, x) = \phi^*(t, x) = \alpha^* \] (29)

or, in terms of energy variables by \( p_t(t, x) = p_t(t, x) = 0 \) and \( \epsilon_t(t, x) = \epsilon_r(t, x) = 0 \). Note that all the configurations in which the beam is not deformed are energetically equivalent. As regard the mass in \( x = L \), the equilibrium configuration will be

\[ q_1^* = q^* + L\alpha^* \] and \[ q_2^* = \alpha^* \] (30)

In conclusion, the energy function \( H_c \) of the controller will be developed in order to regulate the closed-loop system in the configuration \( \chi^* = (q^*, p^*, p_c^*, p_r^*, \epsilon_t^*, \epsilon_r^*) = (q^*, 0, 0, 0, 0, 0, 0) \).

In the remaining part of this section it will proved that, by choosing the controller energy as

\[ H_c(q_c, p_c) := \frac{1}{2} p_c^T M_c^{-1} p_c + \frac{1}{2} K_{c,1} (-q_{c,1} - Lq_{c,2} - q_1)^2 + \frac{1}{2} K_{c,2} (-q_{c,2} - q_2)^2 \]

with \( M_c = M_c^T > 0 \), \( K_{c,1}, K_{c,2} > 0 \) the configuration \( \chi^* \) previously introduced is (globally) asymptotically stable. Since \( q_c \) is given by \cite{28}, we deduce that

\[ H_c(q_c, p_c) := \frac{1}{2} p_c^T M_c^{-1} p_c + \frac{1}{2} K_{c,1} \left( q_1 + \int_D (L - x)\epsilon_r + \epsilon_t \right)^2 + \frac{1}{2} K_{c,2} \left( q_2 + \int_D \epsilon_r - q_2^* \right)^2 \] (31)
If the energy function of the controller is chosen as in (31), then it is possible to stabilize the system in the configuration \( \chi^* \).

As in the case of finite dimensional Hamiltonian system, the stability of a mixed finite and infinite dimensional Hamiltonian system can be proved if it can be shown that the equilibrium is a strict extremum of the total energy of the closed-loop system. The only difference is that, in order to prove the stability for the infinite dimensional part, it is necessary to fix a norm: it is important to note that the stability with respect to this given norm will not assure the stability with respect to a different one. The stability definition in the sense of Lyapunov for mixed finite and infinite dimensional system can be given by the following, [13]:

**Definition 4.2 (Lyapunov stability for mixed systems).** The equilibrium configuration \( \chi^* \) for a mixed finite and infinite dimensional system is said to be stable in the sense of Lyapunov with respect to the norm \( \| \cdot \| \) if for every \( \epsilon > 0 \) there exists \( \delta_\epsilon > 0 \) such that

\[
\| \chi(0) - \chi^* \| < \delta_\epsilon \Rightarrow \| \chi - \chi^* \| < \epsilon
\]

for all \( t > 0 \), where \( \chi(0) \) is the initial configuration of the system.

As proposed in [9] [13], in order to verify the stability of \( \chi^* \), it is necessary to show that \( \chi^* \) is an extremum of the closed-loop energy function \( \mathcal{H}_{cl} \) introduced in (22), with \( H_c \) given by (31), that is the condition \( \nabla \mathcal{H}_{cl}(\chi^*) = 0 \) must hold. Moreover, if \( \Delta \chi \) is the displacement from the equilibrium configuration \( \chi^* \), introduce the non linear functional

\[
\mathcal{N}(\Delta \chi) := \mathcal{H}_{cl}(\chi^* + \Delta \chi) - \mathcal{H}_{cl}(\chi^*)
\]

that is proportional to the second variation of \( \mathcal{H}_{cl} \). Then, the configuration \( \chi^* \) is stable if it is possible to find \( \gamma, \Gamma > 0 \) such that, [13]:

\[
0 \leq \mathcal{N}(\Delta \chi) \leq \Gamma \| \Delta \chi \|^\gamma
\]

(33)

In this case, the norm \( \| \cdot \| \) can be chosen as follows:

\[
\| \Delta \chi \| := \left( |\Delta p|^2 + |\Delta \rho|^2 + |\Delta p_c|^2 + \int_D (\ast \Delta p_t \wedge \Delta p_t + \ast \Delta p_r \wedge \Delta p_r + \ast \Delta \epsilon_t \wedge \Delta \epsilon_t + \ast \Delta \epsilon_r \wedge \Delta \epsilon_r) \right)^{\frac{1}{2}}
\]

with \( | \cdot | \) the usual Euclidean norm. Since

\[
\frac{\partial_p(H + H_c)}{\partial p} = \begin{bmatrix} K_{c,1} (-q_{c,1} - L q_{c,2} - q_{c,1}^2) \\ K_{c,2} (-q_{c,2} - q_{c,2}^2) \end{bmatrix} \]

\[
\frac{\partial_{p, H}}{\partial p} = M^{-1} p_c \quad \delta_{p, H} = \frac{1}{r} p_c \quad \delta_{p, \chi} = \frac{1}{r} x_c
\]

\[
\delta_{\epsilon, (H_c + H)} = [E I \ast \epsilon_r + K_{c,1} (-q_{c,1} - L q_{c,2} - q_{c,2}^2)] (L - x) + K_{c,2} (-q_{c,2} - q_{c,2}^2)
\]

are equal to 0 in \( \chi = \chi^* \), then \( \nabla \mathcal{H}_{cl}(\chi^*) = 0 \). Moreover, also \( \mathcal{H}_{cl}(\chi^*) = 0 \); then, the linear functional introduced in (32) is given by

\[
\mathcal{N}(\Delta \chi) = \frac{1}{2} \Delta p^T M^{-1} \Delta p + \frac{1}{2} \Delta p_c^T M_c^{-1} \Delta p_c + \frac{1}{2} \int_D \left( \frac{1}{r} \ast \Delta p_t \wedge \Delta p_t + \frac{1}{r} \ast \Delta p_r \wedge \Delta p_r + K \ast \Delta \epsilon_t \wedge \Delta \epsilon_t + E I \ast \Delta \epsilon_r \wedge \Delta \epsilon_r \right) + \frac{1}{2} K_{c,1} \left( \Delta q_1 + \int_D (L - x) \Delta \epsilon_r + \Delta \epsilon_t \right)^2 + \frac{1}{2} K_{c,2} \left( \Delta q_2 + \int_D \Delta \epsilon_t \right)^2
\]

Since \( \mathcal{N}(\Delta \chi) \geq 0 \) if \( \Delta \chi \neq 0 \), the stability proof is completed if some constants \( \gamma, \Gamma \) satisfying (33) are found. If \( \Gamma = \max \left\{ \frac{1}{2} M^{-1}, K_{c,1}, K_{c,2}, \frac{1}{2} K, 2 E I \right\} \), then

\[
\mathcal{N}(\Delta \chi) \leq \Gamma \left[ |\Delta p|^2 + |\Delta p_c|^2 + \int_D (\ast \Delta p_t \wedge \Delta p_t + \ast \Delta p_r \wedge \Delta p_r + \ast \Delta \epsilon_t \wedge \Delta \epsilon_t + \ast \Delta \epsilon_r \wedge \Delta \epsilon_r) + \left( \Delta q_1 + \int_D (L - x) \Delta \epsilon_r + \Delta \epsilon_t \right)^2 + \left( \Delta q_2 + \int_D \Delta \epsilon_t \right)^2 \right]
\]
Since
\[
\left( \Delta q_1 + \int_D (x - L) \Delta \varepsilon_r + \Delta \varepsilon_r \right)^2 \leq 2 \Delta q_1^2 + 2L^2 \int_D \star \Delta \varepsilon_r \wedge \Delta \varepsilon_r + 2L \int_D \star \varepsilon_f \wedge \Delta \varepsilon_r \\
\left( \Delta q_2 + \int_D \Delta \varepsilon_r \right)^2 \leq 2 \Delta q_2^2 + 2L \int_D \star \Delta \varepsilon_r \wedge \Delta \varepsilon_r
\]
assuming \( \gamma = 2 \) and \( \Gamma = \tilde{\Gamma} \max \{ 2, 2L + 1, 2L^2 + 2L + 1 \} \), then condition (33) is satisfied and the configuration \( \chi^* \) is stable. Summarizing, the following proposition has been proved.

**Proposition 4.3.** Consider the mixed finite and infinite dimensional port Hamiltonian system of Fig. 2 that is the result of the power conserving interconnection (21) of the subsystems (14), (19) and (20). If the energy function of the controller is chosen according to (31), then the configuration described in (29, 30) is stable in the sense of Lyapunov.

5 Conclusions and future work

Starting from the dpH formulation of the Timoshenko beam, in these notes some considerations about control strategies of the flexible beam have been presented. In particular, the well-known control by damping injection is extended to distributed parameter systems in order to stabilize the beam acting through its boundary and/or its distributed port. Some well-known results already presented in the literature are obtained in this new framework.

Moreover, it has been shown that it is possible to extend the energy shaping by interconnection control technique to treat mixed finite and infinite dimensional systems. In particular, the control of a mechanical system made of a flexible beam with a mass connected at one of its extremity has been presented. The finite dimensional controller, acting on the system through the other extremity, is developed by properly extending the concept of Casimir functions to the infinite dimensional case.

Future work will deal with the extension of these concepts to the modeling and control of simple kinematic chains with flexible links.

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**References**


